Playing Random and Expanding Unique Games

Alexandra Kolla Madhur Tulsiani

Abstract

In this work, we present a spectral algorithm that finds good assignments for instances of Unique Games when the underlying graph has some significant expansion and the constraints are arbitrary Γ-max-lin.

We first analyze the behavior of the SDP by Feige and Lovász [FL92] on random instances of unique games. We show that on random d-regular graphs with permutations chosen at random, the value of the SDP is very small with probability $1 - e^{-\Omega(d)}$. Hence, the SDP provides a proof of unsatisfiability for random unique games. We then give a spectral algorithm for recovering planted solutions. Given a random instance consistent with a given solution on $1 - \epsilon$ fraction of the edges, our algorithm recovers a solution with value $1 - O(\epsilon)$ with high probability at least $1 - e^{-\Omega(d)}$ over the inputs. Using similar arguments as in the planted solution case, we conclude with an algorithm that finds good solutions for a Γ-max-lin expanding unique game. We present both cases in a unified manner in order to emphasize the main ideas that were used in the analysis of the algorithm.

1 Introduction

A unique game is defined in terms of a constraint graph $G = (V, E)$, a set of variables $\{x_u\}_{u \in V}$, one for each vertex u and a set of permutations (constraints) $\Pi_{uv} : [k] \to [k]$, one for each edge (u, v). An assignment to the variables is said to satisfy the constraint on the edge $(u, v) \in E$ if $\pi_{uv}(x_u) = x_v$. The edges are taken to be undirected and hence $\pi_{uv} = (\pi_{vu})^{-1}$. The goal is to assign a value from the set $[k]$ to each variable x_u so as to maximize the number of satisfied constraints.

Khot [Kho02] conjectured that it is NP-hard to distinguish between the cases when almost all the constraints of a unique game are satisfiable and when very few of the constraints are satisfiable. Formally, the statement of the conjecture is the following:

Conjecture 1 *(Unique Games Conjecture) For any constants $\epsilon, \delta > 0$, for any $k > k(\epsilon, \delta)$, it is NP-hard to distinguish between instances of unique games with domain size k where at least $1 - \epsilon$ fraction of constraints are satisfiable and those where at most δ fraction of constraints are satisfiable.*

The Unique Games Conjecture is known to imply optimal inapproximability results for several important problems. For instance, it implies a hardness of approximation within a factor of $2 - \epsilon$ for Vertex Cover [KR03] and within a factor of 0.878 for Max-Cut [KKMO04]. These results are not known to follow from any other complexity assumptions.

Several approximation algorithms using linear and semidefinite programming have been developed for approximating unique games (see [Kho02], [Tre05], [GT06], [CMM06a], [CMM06b]). These algorithms start with an instance where the value of the SDP or LP relaxation is $1-\epsilon$ and round it to a solution with value ν. Here, value of the game refers to the maximum fraction of satisfiable constraints. For $\nu > \delta$, this would give an algorithm to distinguish between the two cases. However, most of these algorithms give good approximations only when ϵ is very small ($\epsilon = O(1/\log n)$ or $\epsilon = O(1/\log k)$) [1]. For constant ϵ however, only the algorithm of [CMM06a] gives interesting parameters with $\nu \approx k^{-\epsilon/(2-\epsilon)}$. We refer the reader to [CMM06a] for a comparison of parameters of various algorithms.

Is is also known (is implicit in [KV05]) that a stronger version of the unique games conjecture, where the underlying constraint graph has significant expansion, would imply hardness of the uniform version of the Sparsest Cut problem. An algorithm for solving unique games on random graphs would thus give partial evidence of a negative answer. In this paper we study the case of random and semi-random permutations, which may help in understanding how (and if) expansion can provide an algorithmic advantage.

Our results

We study the case of random unique games generated by picking a random regular graph of degree d (or a random $G_{n,p}$ graph of average degree d) and picking a random permutation for each edge. We show that with high probability over the choice of instances, the value of the SDP from [FL92] and [Kho02] is at most δ for $d = \Omega(1/\delta^4 + 1/\epsilon^4)$. Here, we think of ϵ, δ as small constants and d as a large constant.

Using techniques from the above analysis, we also study the problem of recovering planted solutions for random unique games and finding good solutions when the given Unique game is a Γ-max-lin expanding instance. Specifically, we start with studying the model where a random instance *consistent with a given solution* is chosen to start with, and an adversary then perturbs ϵ fraction of the constraints. Thus, the given instance has one planted solution with value $1-\epsilon$. We give an algorithm which recovers w.h.p. a solution of value at least $1-O(\epsilon)$ *even when the perturbations are adversarial.* The result for Γ-max-lin expanding constraint graphs follows easily from this analysis.

To obtain both the above results, we analyze the dual of the SDP. We reduce the problem of estimating the value of the SDP to estimating the eigenvalues for an associated matrix M. Since most known eigenvalue analyses are for matrices with independent entries, which does not happen to be the case with M, we adapt the analyses from [BS87] and [AKV02] to our purposes. The planted and expanding Γ-max-lin cases are dealt with by analyzing the eigenvectors of this matrix.

Remark: For the random graphs model, it is possible to prove analogous results in the $G_{n,p}$ model by using the eigenvalue analysis from [FO05]. However, in this model our current estimates only give interesting results in the range $d = \Omega(k^2)$, where $d = pn$ is the expected degree of the constraint graph and k is the size of the alphabet.

[1] It might be good to think of k as $O(\log n)$ since this is range of interest for most reductions

2 Preliminaries

2.1 Unique Games with Γ-max-lin constraints

As noted before, a unique game is defined in terms of a constraint graph $G = (V, E)$, a set of variables $\{x_u\}_{u \in V}$, one for each vertex u and a set of permutations (constraints) $\Pi_{uv} : [k] \to [k]$, one for each edge (u, v). An instance of Unique Games is Γ-max-lin when the constraints are of a very specific form, namely they are all linear equations over some abelian group Γ.

2.2 SDPs and duality

Semidefinite programs are often used as relaxations of 0/1 quadratic programs. In obtaining the relaxations, we often replace 0/1 variables $x_1, \ldots, x_n$ by vectors $\mathbf{v_1}, \ldots, \mathbf{v_n}$. Alternatively, we may think of solving for an $n \times n$ positive semidefinite matrix Y such that $Y_{ij} = \mathbf{v_i} \cdot \mathbf{v_j}$. Then, one way of writing a general SDP is

$$\begin{aligned} \text{maximize} \quad & B \bullet Y \\ \text{subject to} \quad & A_1 \bullet Y = c_1 \\ & A_2 \bullet Y = c_2 \\ & \vdots \\ & A_n \bullet Y = c_n \\ & Y \succeq 0 \end{aligned}$$

where $A_1, A_2, \ldots A_n, B$ are symmetric square matrices and $A \bullet B$ denotes the Frobenius inner product $(= \sum_{i,j} a_{ij} b_{ij})$ of the matrices.Here $Y \succeq 0$ denotes the constraint that Y is positive semidefinite. The dual of the above SDP is

$$\begin{aligned} \text{minimize} \quad & c^T x \\ \text{subject to} \quad & x_1 A_1 + x_2 A_2 \ldots x_n A_n - B \succeq 0 \end{aligned}$$

From (weak) duality, we have that $v_{\text{primal}} \leq v_{\text{dual}}$.

2.3 Spectra of graphs

In the rest of the paper we are going to investigate this SDP by looking at its dual and reducing it to estimating eigenvalues of graphs.

We remind the reader that for a graph G, the adjacency matrix $A = A_G$ is defined as :

$$A_G = \begin{cases} 1 & \text{if } (u, v) \in E \\ 0 & \text{if } (u, v) \notin E \end{cases}$$

If the graph has n vertices, A_G has n real eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \lambda_n$. The eigenvectors that correspond to these eigenvalues form an orthonormal basis of $\mathbb{R}^n$. We note that if the graph is d-regular then the largest eigenvalue is equal to d and the corresponding eigenvector is the all-one's vector.

We can use the Courant-Fisher Theorem to characterize the spectrum of A. The largest eigenvalue satisfies

$$\lambda_1 = \max_{x \in \mathbb{R}^n} \frac{x^T A x}{x^T x}$$

If we denote the first eigenvector by x_1 then

$$\lambda_2 = \max_{x \in \mathbb{R}^n, x \perp x_1} \frac{x^T A x}{x^T x}$$

Similar definitions hold for the eigenvalues λ_i, $i \geq 3$.

3 The dual of Unique-Games SDP for random graphs

We look at the SDP for Unique-Games without the triangle inequality. The SDP is

$$\begin{array}{lcl} \text{maximize} & \sum_{(u,v)\in E} \sum_{i=1}^{k} \mathbf{u}_i \cdot \mathbf{v}_{\pi_{uv}(i)} & \\ \text{subject to} & \mathbf{u}_i \cdot \mathbf{u}_j = 0 & \forall u \in V, \forall i, j \\ & \sum_{i=1}^{k} \|\mathbf{u}_i\|^2 = 1 & \forall u \in V \end{array}$$

The feasible region of the dual can be expressed as $Z \succeq 0$ where Z is an $nk \times nk$ matrix. We use Z_{uv} to denote the $k \times k$ block corresponding to the vertices u and v. The blocks are given by

$$Z_{uv} = \begin{cases} 0 & \text{if } (u,v) \notin E, u \neq v \\ -\frac{1}{2}\Pi_{uv} & \text{if } (u,v) \in E \\ Z_u & \text{if } u = v \end{cases}$$

where Π_{uv} is the permutation matrix corresponding to π_{uv} and Z_u is the (symmetric) matrix of all the variables corresponding to the vertex u. The off-diagonal entries of Z_u are $(Z_u)_{ij} = (Z_u)_{ji} = \frac{1}{2}x^u_{\{i,j\}}$ - a separate variable for each pair $\{i,j\}$ and vertex u. All the diagonal entries are the same, equal to a single variable $x^{(u)}$. The objective function of the whole SDP is $\sum_{u\in V} x^u$.

We will consider dual solutions with $x^u_{\{i,j\}} = 2d/k$ for all $u \in V$ and $i, j \in [k], i \neq j$. Also, we set $x^{(1)} = x^{(2)} = \ldots = x^{(n)} = \lambda + d/2k$. Here λ is taken to be an upper bound on the second eigenvalue. Note that the first eigenvalue of M is d since M can be thought of as the adjacency matrix of a d-regular graph on nk vertices. The objective value as $nd/2k + n\lambda$. Putting in these values for the variables, we will need to show that the following equation is satisfied.

$$\lambda I + \frac{d}{2k} J - \frac{1}{2} M \succeq 0$$

where I is the $nk \times nk$ identity matrix, J is a block diagonal matrix with $k \times k$ blocks of all 1s on the diagonal and M is a block matrix with $M_{uv} = \Pi_{uv}$ if $(u,v) \in E$ and 0 otherwise.

Let z denote the all vector with all coordinates $\frac{1}{\sqrt{nk}}$. Then z is the first eigenvector of M. We prove the following in the next section

Theorem 2 *Let M be a matrix generated according to a random d-regular graph and random permutations on each edge. Then, with probability $1 - e^{-\Omega(d)}$, $\lambda_2(M) \leq Cd^{3/4}$*

Hence, we take $\lambda = Cd^{3/4}$ which is a bound on the second eigenvalue[2]. Note that z is the first eigenvector of both J and M. Since we can express any vector x and $\alpha z + \beta w$ with $w \perp z$, we have

$$\begin{aligned} x^T\left(\lambda I + \frac{d}{2k}J - \frac{1}{2}M\right)x &= (\alpha z + \beta w)^T\left(\lambda I + \frac{d}{2k}J - \frac{1}{2}M\right)(\alpha z + \beta w) \\ &= \lambda + \alpha^2\frac{d}{2} + \beta^2\frac{d}{2}w^T J w - \frac{1}{2}\left(\alpha^2 z^T M z + \beta^2 w^T M w\right) \end{aligned}$$

Since J is positive semidefinite, $z^T M z \leq d$ and $w^T M w \leq Cd^{3/4}$, we have $x^T(\lambda I + \frac{\lambda_1}{2k}J - \frac{1}{2}M) \geq 0$ for every x. This gives that the value of the SDP for random d-regular graphs is $\frac{|E|}{k} + \frac{|E|}{d^{1/4}}$ with high probability.

4 Bounding the second eigenvalue for d-regular graphs

We consider undirected random $2d$-regular graphs G_{2d} on n vertices constructed by choosing d permutations (over n elements) independently at random. For each of the chosen permutations σ and for each vertex u we add to the graph the edge $(u, \sigma(u))$. The unique game is then constructed for by then picking a random permutation π_{uv} (over k elements) for each edge $(u, v) \in E$.

The bound on the second eigenvalue is obtained in two steps. We first by first bound the expected value by examining the trace of a power of the matrix M. We then show a concentration bound using an application of Talagrand's inequality adapted from [AKV02].

4.1 Bounding the mean

In the following argument, it will be convenient to consider the normalized matrices $M^* = (2d)^{-1}M$, $A^* = (2d)^{-1}A$. For any positive integer p, we have $Trace((M^*)^p) = \frac{1}{(2d)^p}Trace(M^p)$ and same for A^*. Let $\rho_1, \rho_2, \cdots, \rho_{nk}$ the eigenvalues of M^* in order of decreasing value. Clearly, $\rho_1 = 1$. Our next goal is to upper-bound the mean value of the quantity $\rho = max\{\rho_2, |\rho_n|\}$. Let p be a large positive integer to be fixed later.

Lemma 3

$$E[\rho] \leq (E[Trace((M^*)^{2p})] - 1)^{1/2p}$$

Proof: Because Trace$((M^*)^{2p}) = \sum_{1\leq i\leq nk} {\rho_i}^{2p}$ and because all the eigenvalues of a symmetric matrix are real, we have :

$$\rho^{2p} \leq Trace((M^*)^{2p}) - 1$$

Taking expectations over the probability space described above,(that is, over all $2d$-regular graphs and over all permutations of k elements within each non-zero block), we have

$$E[\rho] \leq E[\rho^{2p}]^{1/(2p)} \leq (E[Trace((M^*)^{2p})] - 1)^{1/2p}$$

by Jensen's inequality. □

We next relate the value of $E[Trace((M^*)^{2p})]$ to $E[Trace((A^*)^{2p})]$.

[2] We believe that it is possible to improve this bound to even $C\sqrt{d}$ but this is not very important for our purposes.

Claim 4 *Let $A = [a_{ij}]$ be the adjacency matrix of a graph G and M be a block matrix with $M_{uv} = \Pi_{uv}$ if $(u, v) \in E$ and 0 otherwise. Then $E[Trace(M^{2p})] = Trace(A^{2p})$ where p is a positive integer and the expectation on the left hand side is taken over the choice of permutations.*

PROOF: Let S be a set containing all the sequences of $2p+1$ nodes of G that begin and end at the same node. I.e $S = \{uu_1 \cdots u_{2p}u\}$. Each $s \in S$ corresponds to a walk on G of length $2p$ that begins and ends at the same node and therefore also corresponds to a sequence of blocks of the matrix M above that begins and ends at the same block.

For any matrix $Q = [q_{ij}]$ and for any positive integer n we have

$$Trace(Q^n) = \sum_{i_1, i_2, \dots i_n} q_{i_1 i_2} q_{i_2 i_3} \dots q_{i_n i_1}$$

Observe that when Q is the adjacency matrix of a graph, each term in the above sum is 1 if $i_1, i_2, \cdots i_n, i_1$ is a path in the graph and 0 otherwise.

Thus, for the matrices A and M we have

$$\begin{aligned} Trace(A^{2p}) &= \sum_{u_1 u_2 \dots u_{2p} u_1 \in S} a_{u_1 u_2} \dots a_{u_{2p} u_1} \\ Trace(M^{2p}) &= \sum_{\substack{u_1, u_2, \dots, u_{2p} u_1 \in S \\ i_1, i_2, \dots, i_{2p} \in [k]}} m_{(u_1, i_1)(u_2, i_2)} \dots m_{(u_{2p}, i_{2p})(u_1, i_1)} \end{aligned}$$

where the tuple (u, i) corresponds to the index of the ith element of block u.

We can write each term $m_{(u,i)(v,j)} = a_{uv} \dots \mathbb{I}_{\{\pi_{uv}(i)=j\}}$, where the random variable $\mathbb{I}_{\{\pi_{uv}(i)=j\}}$ is 1 when $\pi_{uv}(i) = j$ and 0 otherwise. We can now re-write the trace as

$$Trace(M^{2p}) = \sum_{u_1 u_2 \dots u_{2p} u_1 \in S} a_{u_1 u_2} \dots a_{u_{2p} u_1} \sum_{i_1, i_2, \dots, i_{2p} \in [k]} \mathbb{I}_{\{\pi_{u_1 u_2}(i_1) = i_2\}} \dots \mathbb{I}_{\{\pi_{u_{2p} u_1}(i_{2p}) = i_1\}}$$

and, taking expectation over all permutations

$$E[Trace(M^{2p})] = \sum_{u_1 u_2 \dots u_{2p} u_1 \in S} a_{u_1 u_2} \dots a_{u_{2p} u_1} \sum_{i_1, i_2, \dots, i_{2p} \in [k]} P[\pi_{u_1 u_2}(i_1) = i_2 \wedge \dots \wedge \pi_{u_{2p} u_1}(i_{2p}) = i_1]$$

For multi-indices $U = u_1 u_2 \dots u_{2p}$ and $I = i_1 i_2 \dots i_{2p}$ let $E_{U,I}$ be the event $\{\pi_{u_1 u_2}(i_1) = i_2 \wedge \cdots \wedge \pi_{u_{2p} u_1}(i_{2p}) = i_1\}$. For a fixed U, the events $E_{U,I}$ where I takes all possible values consist of a partition of the whole probability space. Therefore with this notation,

$$E[Trace(M^{2p})] = \sum_U a_{u_1 u_2} \dots a_{u_{2p} u_1} \sum_I P[E_{U,I}] = \sum_U a_{u_1 u_2} \dots a_{u_{2p} u_1} = Trace(A^{2p})$$

□

Hence, to bound ρ, it suffices to bound $E[Trace(A^{2p})]$. The following lemma can be found in [BS87].

Lemma 5 *Let A^* as above and $p = (2 - \epsilon') log_{d/2} n$ a positive integer. Then*

$$E[Trace((A^*)^{2p})] \le \frac{1}{n^{1-\epsilon'}} + 1 + O(\frac{(logn)^4}{n})$$

Claim 6 *Let p be as above. Then for every $\epsilon > 0$ we have the inequality :*

$$E[\rho] \leq (\frac{2}{d})^{1/4}(1 + \epsilon + o(1))$$

PROOF: From claim 4 we have

$$E[Trace((M^*)^{2p})] = E[Trace((A^*)^{2p})]$$

Using lemma 5 we have

$$E[Trace((M^*)^{2p})] \leq (\frac{1}{n^{1-\epsilon'}} + 1 + O(\frac{(logn)^4}{n}))$$

Hence,

$$E[\rho] \leq (E[Trace((M^*)^{2p})] - 1)^{1/2p} = (E[Trace((A^*)^{2p})] - 1)^{1/(2(2-\epsilon')log_{d/2}n)}$$
$$\leq (\frac{1}{n^{1-\epsilon'}})^{\frac{1}{2(2-\epsilon')log_{d/2}n}}(1 + o(1)) = (\frac{2}{d})^{1/4}(1 + \epsilon + o(1))$$

Which follows by the appropriate choice of ϵ'. □

From the above calculations it follows that if λ is the second largest (in absolute value) eigenvalue of M, then

$$E[\lambda] = O(d^{3/4})$$

We note that it is also possible to bound $E[\lambda]$ by $O(\sqrt{d})$ by using the (more involved) bound on $Trace((A^*)^{2p})$ from [Fri91].

4.2 Concentration of λ around the mean

We will next prove that with probability that tends to 1 as $n \to \infty$, λ deviates from its mean by at most $\sqrt{d}$. For that we will first prove concentration of λ around its median, and then use elementary probability techniques to show that the expectation and the median of λ are very close. Namely, we will prove the following theorem :

Theorem 7 *The probability that λ_2 deviates from its median by more than t is at most $4e^{-t^2/128}$. The same estimate holds for the probability that λ_{kn} deviates from its median by more than t. Therefore $Pr[|\lambda - \mu(\lambda)| \geq t] \leq 2e^{-t^2/128}$, where $\mu(\lambda)$ denotes the median of λ.*

For that reason, we will use Talagrand's inequality in a similar manner as in [AKV02].

Theorem 8 *(Talagrand's Inequality) Let $\Omega_1, \Omega_2, \cdots, \Omega_m$ be probability spaces, and let Ω denote their product space. Let $\mathcal{A}$ and $\mathcal{B}$ be two subsets of Ω and suppose that for each $B = (B_1, \cdots, B_m) \in \mathcal{B}$ there is a real vector $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_m)$ such that for every $A = (A_1, \cdots, A_m) \in \mathcal{A}$ the inequality*

$$\sum_{i:A_i \neq B_i} \alpha_i \geq t(\sum_{i=1}^{m} {\alpha_i}^2)^{1/2}$$

holds. Then

$$Pr[\mathcal{A}]Pr[\mathcal{B}] \leq e^{-t^2/4}$$

We now apply Talagrand's inequality to prove theorem 7. We will show the case for λ_2, but the same proof easily carries out for λ_{kn}. Some notation follows:
Let $\binom{m=n+1}{2}$ and consider the product space Ω of the blocks M_{ij},$1 \leq i, j, \leq n$ where each block is a $k \times k$ permutation matrix. We identify each element of Ω with the vector consisting of the corresponding m $k \times k$ blocks. Instead of i, j we will use indices u, v for the block of M corresponding to vertices u, v. Let μ denote the median of λ_2.
Let $\mathcal{A} = \{M | \lambda_2(M) \leq \mu\}$ and $\mathcal{B} = \{M | \lambda_2(M) \geq \mu + t\}$. By definition of the median, $Pr[\mathcal{A}] \geq 1/2$.

For any vector $f = (f(1), \cdots, f(nk)) \in \mathbb{R}^{nk}$ we will denote by $f_i \in \mathbb{R}^k$,$1 \leq i \leq n$ the vector that corresponds to the i-th block of k coordinates of f, i.e. $f_i = (f((i-1)k), f((i-1)k+1), \cdots, f(ik))$. Let $\|f\|$ be the euclidean norm of f.

Proof:(Of theorem 7) Fix a vector $B \in \mathcal{B}$. Let $f^{(1)}, f^{(2)}$ denote the first and second unit eigenvector of B. We define the following cost vector $\alpha = (a_{uv})$ for B.

$$\alpha_{uu} = (\|f_u^{(1)}\| + \|f_u^{(2)}\|)$$

$$\alpha_{uv} = \sqrt{2\alpha_{uu}\alpha_{vv}}, v \neq u$$

Let $D = \{(u,v) | A_{uv} \neq B_{uv}\}$. We will show that

$$\sum_{(u,v)\in D} \alpha_{uv} \geq c \cdot t \cdot (\sum_{1 \leq u \leq v \leq n} \alpha_{uv}^2)^{1/2}$$

Note that

$$\sum_{1 \leq u \leq v \leq n} \alpha_{uv}^2 = (\sum \alpha_{uu})(\sum \alpha_{vv}) = (\|f^{(1)}\|^2 + \|f^{(2)}\|^2)^2 = 4$$

Let $z = c_1 f^{(1)} + c_2 f^{(2)}$ be a unit vector (i.e. ${c_1}^2 + {c_2}^2 = 1$) which is perpendicular to the first eigenvector of A. Note that such a vector can always be found, since the orthogonality of $f^{(1)}$ and $f^{(2)}$ implies that the subspace $span\{f^{(1)}, f^{(2)}\}$ is 2-dimensional. Then

$$z^T A z \leq \lambda_2(A) \leq \mu$$

and

$$z^T A z \geq \lambda_2(B) \geq \mu + t$$

which implies

$$\begin{aligned} t \leq z^T(B-A)z \leq \sum_{(u,v)\in D} {z_u}^T (B_{uv} - A_{uv}) z_v &\leq \sum_{(u,v)\in D : (B_{uv}-A_{uv})_{ij} \neq 0} |z_{ui}||z_{vj}| \\ &\leq \sum_{(u,v)\in D} \sqrt{2\|z_u\|^2}\sqrt{2\|z_v\|^2} \\ &\leq \sum_{(u,v)\in D} 2\sqrt{\alpha_{uu}}\sqrt{\alpha_{vv}} = \sqrt{2} \sum_{(u,v)\in D} \alpha_{uv} \end{aligned}$$

The fourth inequality holds because each coordinate appears at most twice (each block is a permutation matrix). By combining the above, we obtain

$$\sum_{(u,v)\in D} \alpha_{uv} \geq \frac{t}{4\sqrt{2}}(\sum {\alpha_{vv}}^2)^{1/2} \quad \Rightarrow \quad Pr[B] \leq 2e^{\frac{-t^2}{128}}$$

□

We conclude by showing that the eigenvalues are also concentrated around their expectation. Namely,

Theorem 9 $Pr[|\lambda - E[\lambda]| \geq t] \leq e^{-(1-o(1))t^2/128}$

To prove this, we show that the expectation and the median of eigenvalues are very close. We show the result for λ_2 but the result holds for all eigenvalues (with different constants in the exponent).

Claim 10 $E[\lambda_2] - \mu \leq 8\sqrt{2\pi}$

PROOF:

$$E[\lambda_2] - \mu \leq E[|\lambda_2 - \mu|] = \int_0^\infty P[|\lambda_2 - \mu| > t]dt \leq \int_0^\infty 2e^{\frac{-t^2}{128}}dt = 8\sqrt{2\pi}$$

□

5 Recovering solutions by spectral methods

For a given instance of unique games on a graph $G = (V, E)$, let M denote (as before) the $nk \times nk$ symmetric matrix such that the $k \times k$ block M_{uv} is equal to the permutation matrix Π_{uv} if $(u, v) \in E$ and 0 otherwise. We shall now show how the eigenvectors of M may be used to recover good assignments to highly satisfiable instances of unique games in some special cases.

Specifically, we handle the cases when the instances are random regular graphs with random constraints, and also when the constraints are arbitrary Γ-max-lin instances and the underlying graph has some significant expansion. The properties used in both cases are the eigenvalue gap of the underlying graph and small number of eigenvectors of M with high eigenvalue.

We give the analysis for d-regular graphs in both cases to give a unified treatment. While our arguments for random graphs work give better bounds for regular graphs, the ones for expanding Γ-max-lin instances can easily be generalized to non-regular graphs by considering the eigenvectors of the matrix $D-M$ instead of M. Here D denotes an $nk\times nk$ diagonal matrix with $D_{uu} = deg(u)\cdot I$. If we think of M as the adjacency matrix of graph with vertex set $V \times [k]$ and each edge of G replaced by a matching, then $D - M$ can be thought of as the Laplacian matrix of that graph.

We construct an "almost satisfiable" instance according to the following model, which captures both the cases mentioned above:

- Pick a d-regular graph $G = (V, E)$ according to some distribution $\mathcal{D}_G$.
- To every $u \in V$, assign a value $A(u) \in [k]$.

- For every edge $(u,v) \in E$, pick a constraint π_{uv} *consistent with* $A(u)$ *and* $A(v)$ from some distribution $\mathcal{D}_{uv}$. Let M be tha matrix of this completely satisfiable game. We denote the game by (G,k,M).
- Let an adversary pick any $\epsilon|E|$ edges and replace their constraints by arbitrary constraints. Let the new matrix be $\hat{M}$ and let $(G,k,\hat{M})$ denote the perturbed game.

The above model captures the random model with planted solutions if we take $\mathcal{D}_G$ to be the distribution over random d-regular graphs and $\mathcal{D}_{uv}$ to be uniform over permutations consistent with $A(u)$ and $A(v)$. The second case can be realized by taking D_G as any arbitrary distribution over graphs with second eigenvalue (say) at most $(1-\gamma)d$ and $\mathcal{D}_{uv}$ as arbitrary Γ-max-lin constraints.

Let W the span of the eigenvectors of $\hat{M}$ with eigenvalue at least $(1-2\epsilon)d$. The algorithm simply looks at a set $S \subseteq W$ of polynomialy many candidate vectors and "reads-off" an assignment as described below. The set S is chosen differently in each case.

`Recover-Solution`$_S(G,k,\hat{M})$

- For each $x \in S$, construct an assignment A_x by assigning to each vertex u, the index corresponding to the largest entry in the block $(x_{u1},\ldots,x_{uk})$ i.e. $A(u) = \mathrm{argmax}_i x_{ui}$.
- Out of all assignments A_x for $x \in S$, choose the one satisfying the maximum number of constraints.

To choose S, we will look at the analog of W for the matrix M. Let Y denote the span of eigenvectors of M with eigenvalue at least $(1-\epsilon)$. We will first show that if G has a large eigenvalue gap, then every vector in W is close to some vector in Y. We then identify some "nice" vectors in W such that the algorithm works for any vector which close to some nice vector. We then identify a set $S \subseteq W$ such that at least one vector in S is close to a nice vector.

To show that the eigenspaces W and Y are close, we use the following claim which essentially appears in [DK70] as the $\sin\theta$ theorem. We give the proof below for self-containment.

Claim 11 *Let w be a unit length eigenvector of $\hat{M}$ with eigenvalue $\hat{\lambda} \geq (1-2\epsilon)d$ and let λ_s denote the largest eigenvalue of M which is smaller than $(1-2\epsilon)d$. Then, w can be written as $\alpha y + \beta y_\perp$ with $|\beta| \leq \left\|(M-\hat{M})w\right\| / (\hat{\lambda}-\lambda_s)$*

Proof: We have

$$(M-\hat{M})w \;=\; \alpha My + \beta My_\perp - \hat{\lambda}w \;=\; \alpha(My - \hat{\lambda}y) + \beta(My_\perp - \hat{\lambda}y_\perp)$$

Since $(M-\hat{\lambda}I)y$ and $(M-\hat{\lambda}I)y_\perp$ are in orthogonal eigenspaces, we have

$$\left\|(M-\hat{M})w\right\|^2 = \alpha^2\left\|(M-\hat{\lambda}I)y\right\|^2 + \beta^2\left\|(M-\hat{\lambda}I)y_\perp\right\|^2 \geq \beta^2\left\|(M-\hat{\lambda}I)y_\perp\right\|^2$$

However, $\left\|(M-\hat{\lambda})y_\perp\right\| \geq (\hat{\lambda}-\lambda_s)$ which proves the claim. □

Hence, to prove that the space Y does not change by much due to the perturbation, we simply need to bound $\left\|(M-\hat{M})w\right\|$. We shall also need the fact that w is somewhat "uniform" over each block. To formalize this, let $\bar{w}$ be the n-dimensional vector such that $\bar{w}_u = \|w_u\|$ where w_u is the k-dimensional vector $(w_{u1},\ldots,w_{uk})^T$. We then show that $\bar{w}$ is very close to the all-one's vector $\vec{1}$.

Claim 12 *If w is an eigenvector of $\hat{M}$ with eigenvalue more than $(1-2\epsilon)d$ and G has second eigenvalue less than $(1-\gamma)d$, then $\bar{w}$ can be written as $a\vec{1}+b\vec{1}_\perp$ with $|b| \le \sqrt{\frac{2\epsilon}{\gamma}}$*

PROOF: Since, w corresponds to a large eigenvalue, we have that

$$(1-2\epsilon)d \;\le\; (\hat{w})^T\hat{M}\hat{w} \;\le\; \sum_{u,v}\|w_u\|\,A_{uv}\,\|w_v\| \;=\; (\bar{w})^T A\bar{w}$$

Writing $\bar{w}$ as $\frac{a}{\sqrt{n}}\vec{1}+b\vec{1}_\perp$, we get

$$(\bar{w})^T A\bar{w} \;\le\; a^2d + b^2(1-\gamma)d$$
$$\Rightarrow (1-2\epsilon)d \;\le a^2d + b^2(1-\gamma)d \;\Rightarrow\; |b| \le \sqrt{\frac{2\epsilon}{\gamma}}$$

□

Using the above, and the fact that the matrix M is only perturbed in ϵ fraction of the edges, we can now bound $\left\|(M-\hat{M})w\right\|$ as follows.

Claim 13 $\left\|(M-\hat{M})w\right\| \le 5\sqrt{\frac{\epsilon}{\gamma}}$

PROOF: Define the $n\times n$ matrix R as $R_{uv}=1$ when the block $(M-\hat{M})_{uv}$ has any non-zero entries, and $R_{uv}=0$ otherwise. Note that if $(M-\hat{M})_{uv}$ is non-zero, then it must be the difference of two permutation matrices. Thus, for all v $\left\|(M-\hat{M})_{uv}w_v\right\| \le 2R_{uv}\|w_v\|$. We have that

$$\begin{aligned}
\left\|(M-\hat{M})w\right\| \;&=\; \sqrt{\sum_u\left\|\sum_v (M-\hat{M})_{uv}w_v\right\|^2} \;\le\; \sqrt{\sum_u\left(\sum_v\left\|(M-\hat{M})_{uv}w_v\right\|\right)^2}\\
&\le\; \sqrt{\sum_u\left(\sum_v 2R_{uv}\|w_v\|\right)^2}\\
&\le\; 2\|R\bar{w}\|
\end{aligned}$$

To estimate $\|R\bar{w}\|$, we break it up as

$$\|R\bar{w}\| \le \frac{a}{\sqrt{n}}\left\|R\cdot\vec{1}\right\| + b\left\|R\cdot\vec{1}_\perp\right\|$$

Since R has at most d 1s in any row, $b\left\|R\cdot\vec{1}_\perp\right\| \le \sqrt{\frac{2\epsilon}{\gamma}}d$. Also, $\left\|R\cdot\vec{1}\right\| = \sqrt{\sum_u\left(\sum_v R_{uv}\right)^2}$. Since R has a total of ϵnd 1s, this expression is maximized when it has d 1s in ϵn rows. This gives $\frac{1}{\sqrt{n}}\left\|R\cdot\vec{1}\right\| \le \sqrt{\epsilon}d$. Combining with the above, we have that

$$\left\|(M-\hat{M})w\right\| \;\le\; 2\sqrt{\epsilon}d + 2\sqrt{\frac{2\epsilon}{\gamma}}d \;\le\; 5\sqrt{\frac{\epsilon}{\gamma}}$$

□

Combining the above bound with claim 11, we get that any unit-length vector $w\in W$ can be expressed as $\alpha y+\beta y_\perp$ where $y\in Y$ and $|\beta| \le 5\sqrt{\frac{\epsilon}{\gamma}}\cdot\frac{1}{(1-2\epsilon)d-\lambda_s}$. Recall that λ_s was the largest eigenvalue of M smaller than $(1-2\epsilon)d$. We now obtain bounds on λ_s and define the set S of candidate vectors separately for each case.

6 The planted solution model on random graphs

Since G is a d-regular graph and each block of M is a permutation matrix, the first eigenvector of M (with eigenvalue d) is the vector $\frac{1}{nk}\vec{1}$. It is easy to verify that the following vector y is orthogonal to $\vec{1}$ and also has eigenvalue d.

$$y_{ui} = \begin{cases} \frac{k-1}{\sqrt{nk(k-1)}} & \text{if } i = A(u) \\ \\ \frac{-1}{\sqrt{nk(k-1)}} & \text{otherwise} \end{cases}$$

The following claim shows that w.h.p. all other eigenvalues of the matrix M are small and hence y is the only vector orthogonal to $\vec{1}$ with eigenvalue more than $(1-2\epsilon)d$.

Claim 14 *With high probability over the choice of M, $\lambda_i(M) \leq O(\sqrt{d})$ for all $i \geq 3$.*

Proof: Let z be a vector perpendicular to both $\vec{1}$ and y such that $\|z\| = 1$. Then, we must have that

$$\sum_u \sum_i z_{ui} = 0 \quad \text{and} \quad \sum_u \left((k-1) z_{uA(u)} - \sum_{i \neq A(u)} z_{ui} \right) = 0$$

which implies

$$\sum_u z_{uA(u)} = \sum_{i \neq A(u)} z_{ui} = 0$$

We now define z_1 as $(z_1)_{ui} = y_{ui}$ for all $i \neq A(u)$ and $(z_1)_{uA(u)} = 0$. Also, let $z_2 = y - y_1$. Then for every u, $(z_2)_{uA(u)} = z_{uA(u)}$ is the only non-zero coordinate of z_2. Also $\|z_1\|, \|z_2\| \leq 1$. We have,

$$\|Mz\| = \|M(z_1 + z_2)\| = \|Mz_1 + Mz_2\| \leq \|Mz_1\| + \|Mz_2\|$$

However, since all constraints are satisfied by the assignment $x_u = A(u)$, $\|Mz_2\| = \left\|Az_2^G\right\|$, where z_2^G is an n-dimensional "projection" of z_2 on the graph by setting $(z_2^G)_u = (z_2)_u$, and A is the adjacency matrix of the graph. From the above equations we have that $\sum_u z_{uA(u)} = 0$, which means that z_2^G is perpendicular to the first eigenvector of A. Thus, w.h.p.

$$\|Mz_2\| = \left\|Az_2^G\right\| \leq O(\sqrt{d}\left\|z_2^G\right\|) \leq O(\sqrt{d})$$

We now consider a new game with matrix M_{k-1} with alphabet size $k-1$ obtained by deleting the value $A(u)$ for each u. Note that this is a completely random unique game for alphabet size $k-1$, since we chose constraints for M randomly after fixing $\pi_{uv}(A(u)) = A(v)$. Finally, it remains to notice that $\|Mz_1\| = \left\|M_{k-1}z_1^{(k-1)}\right\|$, where $z_1^{(k-1)}$ is the $n(k-1)$-dimensional projection of z_1 obtained by deleting coordinates $z_{uA(u)}$ for all u. We also have

$$\sum_{u,i} (z_1^{(k-1)})_{ui} = \sum_{u, i \neq A(u)} z_{ui} = 0$$

which gives that $z_1^{(k-1)}$ is perpendicular to the first eigenvector of M_{k-1} and hence by the previous eigenvalue estimates,

$$\|Mz_1\| = \left\|M_{k-1}z_1^{(k-1)}\right\| \leq O(\sqrt{d})$$

□

From the above, we get that w.h.p. $\lambda_s \leq O(\sqrt{d})$. Also, if the underlying graph G is random, then its second eigenvalue is $O(\sqrt{d})$ and γ is $1 - o(1)$. Combining this with claims 11 and 13, we see that every vector $w \in W$ can be expressed as $w = \alpha y + \beta y_\perp$, with $|\beta| \leq 6\sqrt{\epsilon}$. Also, this gives $\alpha \geq 1 - 6\sqrt{\epsilon}$.

To choose S, note that $\hat{M}$ has at least one eigenvector orthogonal to $\vec{1}$ with eigenvalue more than $1 - 2\epsilon$, since $y \perp \vec{1}$ and $y^T \hat{M} y \geq (1 - \frac{k-1}{k}\epsilon)d \geq (1 - 2\epsilon)d$. Also, the dimension of W can be at most 2 since every unit vector in W must be close to a unit vector in Y and w.h.p. Y has dimension 2. Let $w \in W$ be the eigenvector of $\hat{M}$ orthogonal to $\vec{1}$. We take $S = \{w, -w\}$.

Also, since $w \perp \vec{1}$, we can express $w = \alpha y + \beta y_\perp$ with both y and $y_\perp$ orthogonal to $\vec{1}$. Then, for one of the vectors w or $-w$, y must be the second eigenvector of the matrix M as described earlier. We now show that the algorithm recovers the corresct assignment to most of the variables.

Claim 15 *Let* $w = \alpha y + \beta y_\perp$ *with* $y_{ui} = (k-1)/\sqrt{nk(k-1)}$ *if* $i = A(u)$ *and* $y_{ui} = -1/\sqrt{nk(k-1)}$ *otherwise. Then, for* ϵ *small enough, the coordinate* $w_{uA(u)}$ *has the maximum value within its block for at least* $(1 - 99\epsilon)n$ *blocks* u *.*

Proof: Within each block u, in order for coordinate $A(u)$ to be no longer the maximum one, it must happen that for some j

$$\alpha \frac{k-1}{\sqrt{nk(k-1)}} + \beta \cdot (w_\perp)_{uA(u)} \leq -\frac{\alpha}{\sqrt{nk(k-1)}} + \beta \cdot (w_\perp)_{uj}$$

This gives

$$\begin{aligned}
& (w_\perp)_{uj} - (w_\perp)_{uA(u)} \geq \frac{k}{\sqrt{nk(k-1)}} \cdot \frac{\alpha}{\beta} \\
\Rightarrow \quad & [(w_\perp)_{uj}]^2 + [(w_\perp)_{uA(u)}]^2 \geq \frac{1}{2}[(w_\perp)_{uj} - (w_\perp)_{uA(u)}]^2 \geq \frac{k}{2n(k-1)} \cdot \frac{\alpha^2}{\beta^2} \\
\Rightarrow \quad & \|(w_\perp)_u\|^2 \geq \frac{k}{2n(k-1)} \cdot \frac{(1-6\sqrt{\epsilon})^2}{36\epsilon}
\end{aligned}$$

.

We call such a block "bad". Assume that there are ηn bad blocks. Then

$$1 \geq \sum_{\text{bad } u} \|(w_\perp)_u\|^2 \geq \eta n \cdot \frac{k}{2n(k-1)} \cdot \frac{(1-6\sqrt{\epsilon})^2}{36\epsilon} \quad \Rightarrow \quad \eta \leq \frac{72\epsilon}{(1-6\sqrt{\epsilon})^2} \leq 99\epsilon$$

□

Therefore, for all but at most 99ϵ fraction of the blocks, the maximum coordinate remains at the same place. The assigment recovered by our algorithm then fails to satisfy at most $99\epsilon nd$ constraints corresponding to these blocks and ϵnd constraints perturbed initially. Thus, the solution violates at most $100\epsilon nd = 200\epsilon|E|$ constraints and has value at least $1 - 200\epsilon$.

7 Expanding instances of Γ-max-lin

In the case of Γ-max-lin, for each edge (u, v) in the graph G, we have a constraint of the form $x_u - x_v = c_{uv}$, where x_u, x_v are variables taking values in Γ and $c_{uv} \in \Gamma$. Let k denote the size of the group Γ. As before, we consider a matrix $\hat{M}$ for the given instance, and think of it as an adversarial perturbation on ϵ-fraction of the edges of another matrix M corresponding to a fully satisfiable instance. Let A be an assignment such that the values $x_u = A(u)$ satisfy all the constraints in the instance corresponding to M.

As in the previous analysis, we assume that the graph is d-regular with second eigenvalue at most $d(1-\gamma)$. We will be able to distinguish instances of Γ-max-lin in which $(1-\epsilon)$ fraction of the constraints are satisfiable from those in which at most δ fraction of the cosntraints are satisfiable for $\gamma = \Omega(\epsilon^{1/3})$.

For the matrix M, we define the eigenvectors $y^{(0)}, \ldots, y^{(k-1)}$ as

$$y_{ui}^{(s)} = \begin{cases} \frac{1}{\sqrt{n}} & \text{if} \quad i = A(u) + s \mod k \\ 0 & \text{otherwise} \end{cases}$$

Note that for Γ-max-lin, if $\forall u : \; x_u = A(u)$ is a satisfying assignment, then so is $\forall u : \; x_u = A(u) + s$. Hence, the vectors $y^{(0)}, \ldots, y^{(k-1)}$ correspond to satisfying assignments and are eigenvectors with eigenvalue d for the matrix M. We now show that any eigenvector which is orthogonal to all these vectors has eigenvalue at most $d(1-\gamma)$.

Claim 16 *Let x be a vector such that $x \perp y^{(s)} \forall s$. Then $x^T M x \leq (1-\gamma)d$.*

Proof: Since $x \perp y^{(s)} \forall s$, we have

$$\forall s \in \{0, \ldots, k-1\} \quad \sum_u x_{uA(u)+s} = 0$$

We then decompose x into $x^0, \ldots, x^{(k-1)}$, where

$$x_{ui}^{(s)} = \begin{cases} x_{ui} & \text{if} \quad i = A(u) + s \mod k \\ 0 & \text{otherwise} \end{cases}$$

It is immediate from the definition that $x = \sum_s x^{(s)}$ and that $\|x\|^2 = \sum_s \left\|x^{(s)}\right\|^2$. To bound the eigenvalue corresponding to x, note that

$$x^T M x = \sum_{s,t} (x^{(s)})^T M x^{(t)}$$

Let e_i denote the ith unit vector in k-dimensions. We can then write $x_u^{(s)}$ as $x_{uA(u)+s} e_{A(u)+s}$. Using this notation, we compute the terms in the above equation as

$$(x^{(s)})^T M x^{(t)} = \sum_{(u,v)\in E} (x_u^{(s)})^T \Pi_{uv} (x_v^{(t)}) = \sum_{(u,v)\in E} x_{uA(u)+s} x_{vA(v)+t} \cdot (e_{A(u)+s})^T \Pi_{uv} e_{A(v)+t}$$

Since the permutation maps $A(u)$ to $A(v)$ and $A(u)+s$ to $A(v)+s$ for all s, $(x^{(s)})^T M x^{(t)} = 0$ when $s \neq t$. For the rest of the terms, we have

$$(x^{(s)})^T M x^{(s)} \;=\; \sum_{(u,v)\in E} x_{uA(u)+s} x_{vA(v)+s} \;\leq\; d(1-\gamma) \left\|x^{(s)}\right\|^2 \qquad (\text{Since } \sum_u x_{uA(u)+s} = 0)$$

Hence,

$$x^T M x \;=\; \sum_s (x^{(s)})^T M x^{(s)} \;\leq\; d(1-\gamma) \sum_s \left\|x^{(s)}\right\|^2 \;=\; d(1-\gamma)\left\|x\right\|^2$$

□

We take Y to be the span of $y^{(0)}, \ldots, y^{(k-1)}$. From the above, we know that the next eigenvalue smaller than $(1-2\epsilon)d$ for M is $\lambda_s \leq (1-\gamma)d$. Note that for all $s \in \{0, \ldots, k-1\}$, we have $(y^{(s)})^T \hat{M} y^{(s)} \geq d(1-\epsilon)$. Let w be any unit-length eigenvector of $\hat{M}$, with eigenvalue at least $(1-2\epsilon)d$. By claims 11 and 13, we can express w as $\sum_s \alpha_s y^{(s)} + \beta y_\perp$ with

$$|\beta| \;\leq\; 5\sqrt{\frac{\epsilon}{\gamma}} d \cdot \frac{1}{(1-2\epsilon)d - \lambda_s} \;\leq\; 5\sqrt{\frac{\epsilon}{\gamma}} \cdot \frac{1}{\gamma - 2\epsilon} \;\leq\; 6\sqrt{\frac{\epsilon}{\gamma^3}}$$

Note that this also implies that the eigenspace of vectors with eigenvalue greater than $(1-\epsilon)d$ has dimension at most k (otherwise we would find a vector othogonal to $y^{(0)}, \ldots, y^{(k-1)}$ which cannot be close to their span).

Hence, for $\gamma = \Omega(\epsilon^{1/3})$, the eigenspace of the first k eigenvectors of $\hat{M}$ (W) is close to the eigenspace of the first k eigenvectors of M (Y). Also, Y contains the vectors $y^{(0)}, \ldots, y^{(k-1)}$ which encode the solutions. As in claim 15 we can show that the algorithm works for any vector close to one of the vector $y^{(s)}$.

Claim 17 *If x is a vector such that $v = \alpha y^{(s)} + \beta y_\perp$ for some $y^{(s)}$ with $\alpha > 0$, then the coordinate $x_{uA(u)+s}$ is maximum in at least $(1 - \frac{\beta^2}{\alpha^2} n)$ blocks.*

Proof: Within each block u, in order for coordinate $A(u)+s$ to be no longer the maximum one, it must happen that for some j

$$\alpha \frac{1}{\sqrt{n}} \leq \beta \cdot (y_\perp)_{uj}$$

However,this gives

$$\|(y_\perp)_u\| \;\geq\; (y_\perp)^2_{uj} \;\geq\; \frac{\alpha^2}{\beta^2 n}$$

Since $\|y_\perp\| = 1$, this can only happen for at most $\frac{\beta^2}{\alpha^2} n$ blocks. □

To find a vector v close to one of the vectors $y^{(s)}$, we discretize the eigenspace of the first k eigenvectors of $\hat{M}$. Let $w^{(0)}, \ldots, w^{(k-1)}$ be the eigenvectors. We define the set S as

$$S \;=\; \left\{ v = \sum_{s=0}^{k-1} \alpha_s w^{(s)} \;\middle|\; \alpha_s \in \frac{1}{10\sqrt{k}}\mathbb{Z},\; \|v\| \leq 1 \right\}$$

S contains at least one vector v such that $v = \alpha y^{(s)} + \beta y_\perp$ for some s and $\beta \leq 1/10 + 6\sqrt{\epsilon/\gamma^3} < 1/5$ for $\gamma > 20\epsilon^{1/3}$. Thus, for this vector v, $\texttt{Recover-Solution}_S(G, k, \hat{M})$ recovers an assignment which agrees with $y^{(s)}$ in $(1 - \frac{1}{24})$ fraction of the block. Hence, the assignment violates at most $\frac{1}{24}nd + \epsilon nd < nd/20$ constaints. Since the total number of constraints is $nd/2$, this satisfies more than 90 percent of the constraints.

Finally, it remains to argue that the running time of the algorithm is polynomial. It can be calculated (see, for instance [FO05]) that the number of points in the set S is at most $e^{k \ln 90}$. Since $k = O(\log n)$ (this must hold for the long-code based reductions to be polynomial time), the number of points is polynomial in n. Hence, the algorithm runs in polynomial time.

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