On the Expansion of Group–based Lifts

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Abstract

A k-lift of an n-vertex base graph G is a graph H on $n \times k$ vertices, where each vertex v of G is replaced by k vertices $v_1, \cdots, v_k$ and each edge (u, v) in G is replaced by a matching representing a bijection π_{uv} so that the edges of H are of the form $(u_i, v_{\pi_{uv}(i)})$. Lifts have been studied as a means to efficiently construct expanders. In this work, we study lifts obtained from *groups and group actions*. We derive the spectrum of such lifts via the representation theory principles of the underlying group. Our main results are:

1. There is a constant c_1 such that for every $k \geq 2^{c_1 nd}$, there does not exist an Abelian k-lift H of any n-vertex d-regular base graph such that H is *almost Ramanujan* (nontrivial eigenvalues of the adjacency matrix at most $\mathcal{O}(\sqrt{d})$ in magnitude). This can be viewed as an analog of the well-known no-expansion result for Abelian Cayley graphs.

2. A uniform random lift in a cyclic group of order k of any n-vertex d-regular base graph G, with the nontrivial eigenvalues of the adjacency matrix of G bounded by λ in magnitude, has the new nontrivial eigenvalues also bounded by $\lambda + \mathcal{O}(\sqrt{d})$ in magnitude with probability $1 - ke^{-\Omega(n/d^2)}$. In particular, there is a constant c_2 such that there exists a lift of a Ramanujan graph in a cyclic group of order k which is almost Ramanujan, for every $k \leq 2^{c_2 n/d^2}$. We use this fact to design a quasi-polynomial time algorithm to construct almost Ramanujan expanders deterministically.

The existence of expanding lifts in cyclic groups of order $k = 2^{\mathcal{O}(n/d^2)}$ can be viewed as a lower bound on the order k_0 of the largest cyclic group that produces expanding lifts. Our two results show that the lower bound closely matches the upper bound for k_0 (upto a factor of d^3 in the exponent), thus suggesting a threshold phenomenon. We believe that our results could prove crucial in constructing families of almost Ramanujan expanders of all degrees in polynomial time.

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1 Introduction

Expander graphs have spawned research in pure and applied mathematics during the last several years, with several applications to multiple fields including complexity theory, the design of robust computer networks, the design of error-correcting codes, de-randomization of randomized algorithms, compressed sensing and the study of metric embeddings. For a comprehensive survey of expander graphs see [Sar06, HLW06].

Informally, an expander is a graph where every small subset of the vertices has a relatively large edge boundary. Most applications are concerned with sparse d-regular graphs G, where the largest eigenvalue of the adjacency matrix A_G is d. In case of a bipartite graph, the largest and smallest eigenvalues of A_G are d and $-d$, which are referred to as trivial eigenvalues. The expansion of the graph is related to the difference between d and λ, the first largest (in absolute value) non-trivial eigenvalue of A_G. Roughly, the smaller λ is, the better the graph expansion. The Alon-Boppana bound ([Nil91]) states that $\lambda \geq 2\sqrt{d-1} - o(1)$, thus graphs with $\lambda \leq 2\sqrt{d-1}$ are optimal expanders and are called Ramanujan.

A simple probabilistic argument can show the existence of infinite families of expander graphs [Pin73]. However, constructing such infinite families explicitly has proven to be a challenging and important task. It is easy to construct Ramanujan graphs with a small number of vertices: d-regular complete graphs and complete bipartite graphs are Ramanujan. The challenge is to construct an infinite family of d-regular graphs that are all Ramanujan, which was first achieved by Lubotzky, Phillips and Sarnak [LPS88] and Margulis [Mar88]. They built Ramanujan graphs from Cayley graphs. All of their graphs are regular, have degrees $p+1$ where p is a prime, and their proofs rely on deep number theoretic facts. In two recent breakthrough papers, Marcus, Spielman, and Srivastava showed the existence of bipartite Ramanujan graphs of all degrees [MSS13, MSS15]. However their results do not provide an efficient algorithm to construct those graphs. A striking result of Friedman [Fri08] and a slightly weaker but more general result of Puder [Pud13], shows that almost every d -regular graph on n vertices is very close to being Ramanujan i.e. for every $\epsilon > 0$, asymptotically almost surely, $\lambda < 2\sqrt{d-1} + \epsilon$. It is still unknown whether the event that a random d-regular graph is exactly Ramanujan happens with constant probability. Despite the large body of work on the topic, all attempts to efficiently construct large Ramanujan expanders of any given degree have failed, and exhibiting such constructions remains an intriguing open problem.

A combinatorial approach to constructing expanders, initiated by Friedman [Fri03], is to prove that one may obtain new (larger) Ramanujan graphs from smaller ones. In this approach, one starts with a base graph G which one "lifts" to obtain a larger graph H. More concretely, a k-lift of an n-vertex base-graph G is a graph H on $k \times n$ vertices , where each vertex u of G is replaced by k vertices $u_1, \cdots, u_k$ and each edge (u, v) in G is replaced by a matching between $u_1, \cdots, u_k$ and $v_1, \cdots, v_k$. In other words, for each edge (u, v) of G there is a permutation π_{uv} so that the corresponding k edges of H are of the form $(u_i, v_{\pi_{uv}(i)})$. The graph H is a (uniformly) *random* lift of G if for every edge (u, v) the bijection π_{uv} is chosen uniformly and independently at random from the set of permutations of k elements, S_k.

Since we are focusing on Ramanujan graphs, we will restrict our attention to lifts of d-regular graphs. It is easy to see that any lift H of a d-regular base-graph G is itself d-regular and inherits all the eigenvalues of G (which, hereafter we refer to as "old" eigenvalues, whereas the rest of the eigenvalues are referred to as "new' eigenvalues'). In order to use lifts for building expanders, it is necessary that the lift would also inherit the expansion properties of its base graph. One hopes that a random lift of a Ramanujan graph will also be (almost) Ramanujan with high probability or even that there exists a k-lift which is (almost) Ramanujan for some bounded k. **(Karthik:** last part of the statement is unclear to me.) ??

Friedman [Fri03] first studied the eigenvalues of random k-lifts of regular graphs and proved that every new eigenvalue of H is $\mathcal{O}(d^{3/4})$ with high probability. He conjectured a bound of $2\sqrt{d-1}+o(1)$, which would be tight (see, e.g. [Gre95]). Linial and Puder [LP10] improved Friedman's bound to $\mathcal{O}(d^{2/3})$. Lubetzky, Sudakov and Vu [LSV11] showed that the absolute value of every nontrivial eigenvalue of the lift is $\mathcal{O}(\lambda \log d)$, where λ is the second largest (in absolute value) eigenvalue of the base graph, improving on the previous results when G is significantly expanding. Adarrio-Berry and Griffiths [ABG10] further improved the bounds above by showing that every new eigenvalue of H is $\mathcal{O}(\sqrt{d})$, and very recently, Puder [Pud13] proved the nearly-optimal bound of $2\sqrt{d-1} + 1$. All those results hold with probability tending to 1 as $k \to \infty$, thus the order k of the lift in question needs to be large. Nearly no results were known in the regime where k is bounded with respect to the number of nodes n of the graph.

Bilu and Linial [BL06] were the first to study k-lifts of graphs with bounded k, and suggested constructing

Ramanujan graphs through a sequence of 2-lifts of a base graph: start with a small d-regular Ramanujan graph on some finite number of nodes (e.g. K_{d+1}). Every time the 2-lift operation is performed, the size of the graph doubles. If there is a way to preserve expansion after lifting, then repeating this operation will give large good expanders of the same bounded degree d. The authors in [BL06] showed that if the starting graph G is significantly expanding so that $|\lambda(G)| = \mathcal{O}(\sqrt{d \log d})$, then there exists a random 2-lift of G that has all its new eigenvalues upper-bounded in absolute value by $\mathcal{O}(\sqrt{d \log^3 d})$. In the recent breakthrough work of Marcus, Spielman and Srivastava [MSS13], the authors showed that for every bipartite graph G, there exists a 2-lift of G, such that the new eigenvalues achieve the Ramanujan bound of $2\sqrt{d-1}$, but their result still does not provide any efficient algorithm to find such lifts.

1.1 Our Results

In this work, we study lifts as a means to efficiently construct almost Ramanujan expanders of all degrees. We derive these lifts from groups. This is a natural generalization of Cayley graphs.

Definition 1 (Γ-lift)**.** Let Γ be a group of order k with $\cdot$ denoting the group operation. A Γ-lift of an n-vertex base graph $G(V, E)$ is a graph $H = (V \times \Gamma, E')$ obtained as follows: it has $k \times n$ vertices, where each vertex u of G is replaced by k vertices $\{u\} \times \Gamma$. For each edge (u, v) of G, we choose an element $g_{u,v} \in \Gamma$ and replace that edge by a perfect matching between $\{u\} \times \Gamma$ and $\{v\} \times \Gamma$ that is given by the edges (u_i, v_j) for which $g_{u,v} \cdot i = j$.

We denote $|\Gamma| = k$ to be the order of the lift. We refer to Γ-lifts obtained using $\Gamma = \mathbb{Z}/k\mathbb{Z}$, the additive group of integers modulo k, as shift k-lifts. Since every cyclic group of order k is isomorphic to $\mathbb{Z}/k\mathbb{Z}$, we have that Γ-lifts are shift k-lifts whenever Γ is a cyclic group.

A tight connection between the spectrum of Γ-lifts and the representation theory of the underlying group Γ is known [MS95, FKL04]. This connection tells us that the lift graph incurs the eigenvalues of the base graph, while its new eigenvalues are the union of eigenvalues of a collection of matrices arising from the irreducible representations of the group and the group elements assigned to the edges. This connection has been recently used in [HPS15] in the context of expansion of lifts, aiming to generalize the results in [MSS15]. In order to understand the expansion properties of the lifts, we focus on the new eigenvalues of the lifted graph. We address the expansion of Γ-lifts obtained from cyclic groups and abelian groups.

We present a high probability bound on the expansion of random shift k-lifts for bounded k. Our results for 2-lifts and more generally, for shift k-lifts are as follows:

Theorem 1. *Let G be a d -regular graph with non-trivial eigenvalues at most λ in absolute value, and H be a uniformly random 2-lift of G. Let λ_{new} be the largest new eigenvalue of H in magnitude. Then*

$$\lambda_{new} = \mathcal{O}(\lambda)$$

with probability $1 - e^{-\Omega(n/d^2)}$. Moreover, if G is moderately expanding such that $\lambda \leq \frac{d}{\log d}$, then

$$\lambda_{new} - \lambda = \mathcal{O}(\sqrt{d})$$

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Theorem 2. *Let G be a d -regular graph with non-trivial eigenvalues at most λ in absolute value, and H be a random shift k-lift of G. Let λ_{new} be the largest new eigenvalue of H in magnitude. Then*

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with probability $1 - k \cdot e^{-\Omega(n/d^2)}$. Moreover, if G is moderately expanding such that $\lambda \leq \frac{d}{\log d}$, then

$$\lambda_{new} - \lambda = \mathcal{O}(\sqrt{d})$$

with probability $1 - k \cdot e^{-\Omega(n/d^2)}$.

In particular, if we start with G being a Ramanujan expander, then w.h.p. a random shift k-lift will be almost Ramanujan, having all its new eigenvalues bounded by $\mathcal{O}(\sqrt{d})$.

Remark 1. In contrast to the case of lifts of order $k \to \infty$, the dependency on λ is necessary for bounded k. This has previously been observed by the authors in [BL06] who gave the following example: Let G be a disconnected graph on n vertices that consists of $n/(d+1)$ copies of K_{d+1}, and let H be a random 2-lift of G. Then the largest non-trivial eigenvalue of G is $\lambda = d$ and it can be shown that with high probability, $\lambda_{new} = \lambda = d$. Therefore, our eigenvalue bounds are nearly tight.

Remark 2. Our result for 2-lifts improves upon the $\log d$ factor present in the result of Bilu-Linial [BL06]. This factor arises in their analysis due to the use of the converse of the Expander Mixing Lemma (EML) along with an ϵ-net style argument. The converse EML is provably tight, so straightforward use of the converse EML will indeed incur the $\log d$ factor. We are able to improve the eigenvalue bound by performing a deeper analysis of the ϵ-net argument, avoiding direct use of the converse EML. **(Karthik:** Bilu-Linial also do an ϵ-net type argument; changed the remark. Please verify.) ??

Lifts based on groups immediately suggest an algorithm towards building d-regular n-vertex Ramanujan expanders. In order to describe this algorithm, we first recall the brute-force algorithm that follows from the existential result of [MSS13]. The idea is to start with the complete bipartite graph $K_{d,d}$ and lift the graph $\log_2(n/2d)$ times. At each stage, brute force searching over the space of all possible 2-lifts and picking the best (most expanding) one. However, since a graph (V, E) has $2^{|E|}$ possible 2-lifts, it follows that the final lift will be chosen from among $2^{nd/4}$ possible 2-lifts, which means that the brute force algorithm will run in time exponential in nd.

Next, suppose that for every $k \geq 2$, we are guaranteed the existence of a group Γ of order k such that for every base graph there exists a Γ-lift that has all its new eigenvalues at most $2\sqrt{d-1}$ in absolute value (e.g., suppose for every k and for every base graph, there exists a shift k-lift that has all new eigenvalues with absolute value at most $2\sqrt{d-1}$); then a brute force algorithm similar to the one above, would perform only one lift operation of the base graph $K_{d,d}$ to create a Γ-lift with $n = 2dk$ vertices. This algorithm would only have to choose the best among k^{d^2} possibilities (k different choices of group element per edge of the base graph), which is polynomial in n, the size of the constructed graph. Here we have assumed that d is a constant. This motivates the following question: what is the largest possible group Γ that might produce expanding Γ-lifts? Our next result rules out the existence of large abelian groups that might lead to (even slightly) expanding lifts.

Theorem 3. *For every n-vertex d-regular graph G, $\epsilon \in (0, 1)$, and abelian group Γ of size at least*

$$k = exp\left(\frac{nd \log \frac{1}{\epsilon} + \log n}{\log \frac{1}{e\epsilon}}\right),$$

all Γ-lifts of G have second largest eigenvalue at least ϵd. In particular, when $k = 2^{\Omega(nd)}$, there is no Γ-lift H of any n-vertex d-regular graph G with $\lambda(H) = \mathcal{O}(\sqrt{d})$ whenever Γ is an abelian group of order k.

Remark 3. The first and only known efficient constructions of Ramanujan expanders are Cayley graphs of certain groups [LPS88]. We observe that a Cayley graph for a group Γ with generator set S can be obtained as a Γ-lift of the bouquet graph (a graph that consists of one vertex with multiple self loops) [Mak15]. Our no-expansion result for abelian groups complements the known result on no-expansion of abelian Cayley graphs [FMT06].

Remark 4. Our Theorems 3 and 2 can be viewed as lower and upper bounds on the largest order k_0 of a group Γ such that for every n-vertex graph, there exists a Γ-lift for which all the new eigenvalues are small. On the one hand, Theorem 2 shows that, for $k = 2^{\mathcal{O}(n/d^2)}$, most of the shift k-lifts of a Ramanujan graph have their new eigenvalues upper-bounded by $\mathcal{O}(\sqrt{d})$. On the other hand, Theorem 3 shows that for $k \geq 2^{\Omega(nd)}$, there is no shift k-lift that achieves such expansion guarantees. This suggests a threshold behaviour for k_0.

Moreover, Theorem 2 leads to a deterministic quasi-polynomial time algorithm for constructing almost Ramanujan (with $\lambda = \mathcal{O}(\sqrt{d})$) families of graphs.

Theorem 4. *There exists an algorithm to construct a d-regular n-vertex graph G such that $\lambda(G) = \mathcal{O}(\sqrt{d})$ in $2^{\mathcal{O}(d^4 \log^2 n)}$ time.*

Algorithm 1 Quasi-polynomial time algorithm to construct expanders of arbitrary size n

1: Pick an r such that $2^{cr/d^2} \cdot r = n$, for a constant c given by Theorem 2. Do an exhaustive search to find a d-regular graph G' on r vertices with $\lambda = \mathcal{O}(\sqrt{d})$.
2: For $k = 2^{cr/d^2}$, do an exhaustive search to find a shift k-lift G of G' with minimum $\lambda(G)$.

Proof of Theorem 4. We use Algorithm 1. We note that the choice of r in the first step ensures that $r = \mathcal{O}(d^2 \log n)$. By Theorem 2, there exists a lift G of G' such that $\lambda(G) = \mathcal{O}(\sqrt{d})$. Thus, the exhaustive search in the second step gives a graph G with $\lambda(G) = \mathcal{O}(\sqrt{d})$.

For the running time, we note that the first step can be implemented to run in time $2^{\mathcal{O}(r^2)} = 2^{\mathcal{O}(d^4 \log^2 n)}$. To bound the running time of the second step, we observe that for each edge in G', there are k possible choices. Therefore the total search space is at most $k^{rd/2} = 2^{cr^2/d} = 2^{\mathcal{O}(d^3 \log^2 n)}$ and for each k-lift, it takes *poly*(n) time to compute $\lambda(G)$. Thus, the overall running time of the algorithm is $2^{\mathcal{O}(d^4 \log^2 n)}$.

□

Organization. We give some preliminary definitions, notations, facts and lemmas in Section 2. In particular, we recall the tight connection between the spectrum of Γ-lifts and the representation of Γ in Section 2.4. We prove Theorem 3 in Section 3. For the purpose of intuition, we present and prove a slightly weaker version of Theorem 1 (see Theorem 11) in Section 4. We prove the concentration inequality (Lemma 3) needed for the weaker version in Section 5. We use a stronger version of the concentration inequality and prove Theorems 1 and 2 in Section 6.

2 Preliminaries

2.1 Notations

Let $G = (V, E)$ be a graph with vertex set V, $|V| = n$ and edge set E. Let A be the adjacency matrix of the graph and let $\lambda_1 \geq \lambda_2 \geq \dots \lambda_n$ be its n eigenvalues. Let $\lambda(G) = \max_{i:[2,n]} |\lambda_i|$. Note that since A is a real, symmetric matrix its eigenvalues are also real. Moreover if G is regular with degree d it is well-known that $\lambda_1 = d$ and that $\lambda(G) \leq d$. If G is bipartite, then $\lambda_n = -d$ and we define $\lambda(G) = \max_{i:[2,n-1]} |\lambda_i|$. Throughout the paper, G will be a d-regular graph and we will be concerned with eigenvalues of adjacency matrices.

For any two subsets $S, T \subseteq V$ let $E(S, T)$ be the number of edges between S and T. For a matrix M, we denote by $||M||$ its spectral radius. For a vector x the set $S(x)$ denotes its support, i.e. the set of coordinates of x with a non-zero value. We define log() to be the log function with base 2. We represent e^x by $exp(x)$.

Given a vector $x \in \{0, \pm 1/2, \pm 1/4 \dots\}$ we define the *diadic decomposition* of x as the set $\{2^{-i} u_i\}$ where each u_i defined as

$$[u_i]_j = \begin{cases} 1, & \text{if } x_j = 2^{-i} \\ -1, & \text{if } x_j = -2^{-i} \\ 0, & \text{otherwise} \end{cases}$$

We use the following combinatorial identities.

Lemma 1 (Discretization Lemma)**.** *For any $x \in \mathbb{R}^n$,$||x||_\infty \leq 1/2$ and M such that the diagonal entries of M are 0, there exists $y \in \{\pm 1/2, \pm 1/4, \dots\}^n$ such that $|x^T M x| \leq |y^T M y|$ and $\|y\|^2 \leq 4\|x\|^2$. Moreover, each entry of x between $\pm 2^{-i}$ and $\pm 2^{-i-1}$ is rounded to either $\pm 2^{-i}$ or $\pm 2^{-i-1}$.*

Similarly, for any $x_1, x_2 \in \mathbb{R}^n$, $||x_1||_\infty, ||x_2||_\infty \leq 1/2$, there exists $y_1, y_2 \in \{\pm 1/2, \pm 1/4, \dots\}^n$ such that $|x_1^T M x_2| \leq |y_1^T M y_2|$,$\|y_1\|^2 \leq 4\|x_1\|^2$,$\|y_2\|^2 \leq 4\|x_2\|^2$ and each entry of x_1, x_2 between 2^{-i} and 2^{-i-1} is rounded to either 2^{-i} or 2^{-i-1}.

Proof of Lemma 1. To obtain such a vector y we take a vector x and round its coordinates independently with the following probabilistic rule. Let $x_i = \pm(1+\delta_i)2^{-i}$ be the i^{th} coordinate of x. We round x_i to $sign(x_i) \cdot 2^{-i+1}$ with probability δ_i and $sign(x_i) \cdot 2^{-i}$ with probability $1-\delta_i$. Let the rounded vector be x'. Note that $E[x'_i] = x_i$. Now since each coordinate is rounded independently and the diagonal entries of M are 0, we get that $E[x'^T M x'] = x^T M x$. This implies there exists a $y \in \{\pm 1/2, \pm 1/4, \ldots\}^n$ that can be generated by this rounding such that $|x^T M x| \leq |y^T M y|$. Also it is easy to see that $\|y\|^2 \leq 4\|x\|^2$ and by definition every coordinate in y with value between $\pm 2^{-i}$ and $\pm 2^{-i-1}$ is rounded to either $\pm 2^{-i}$ or $\pm 2^{-i-1}$. The proof of the second part of the lemma is the same as the first part. Here we obtain x'_1 and x'_2 by the same procedure and follow the same argument to get y_1 and y_2. $\square$

Lemma 2. *Assuming that $r^t \leq z/2$, $r \geq 2$, $x > 1/2$, we have the following inequality:*

$$\sum_{i=0}^{i=t} (r^i \log(z/r^i))^x \leq c(r)(r^t \log(z/r^t))^x$$

where $c(r)$ is a constant depending only on r and $c(2) < 9$.

Proof of Lemma 2. For all i define $a_i = (r^i \log(z/r^i))^x$. Let us consider the ratio of consecutive terms a_{i+1}/a_i for $i \in [0, t-1]$.

$$\begin{aligned}
\frac{a_{i+1}}{a_i} &= \left(\frac{r^{i+1}\log(z/r^{i+1})}{r^i \log(z/r^i)}\right)^x \\
&= \left(r\left(1 - \frac{\log(r)}{\log(z) - i\log(r)}\right)\right)^x \\
&\geq \left(r\left(1 - \frac{\log(r)}{1 + (t-i)\log(r)}\right)\right)^x && (r^t \leq z/2)
\end{aligned}$$

If $i \leq t-2$, we get that $a_{i+1}/a_i \geq r^x \left(\frac{1+\log(r)}{1+2\log(r)}\right)^x = \alpha(r)$. It is easy to see that $\alpha(r) > \frac{2}{\sqrt{3}} > 1$ for $r \geq 2$. Also for $i = t-1$, we get that $a_{i+1}/a_i \geq (r/(1+\log(r)))^x \geq 1$. Now consider the sum S_{-1} defined as

$$\begin{aligned}
& S_{-1} = a_0 + a_1 + \ldots + a_{t-1} \\
& \Rightarrow \alpha(r) S_{-1} = \alpha(r)(a_0 + a_1 + \ldots + a_{t-1}) \\
& \Rightarrow (\alpha(r) - 1) S_{-1} = -a_0 + (\alpha(r)a_0 - a_1) + (\alpha(r)a_1 - a_2) \ldots + a_{t-1}\alpha(r) \\
& \Rightarrow (\alpha(r) - 1) S_{-1} \leq a_{t-1}\alpha(r) && (a_{i+1} \geq \alpha(r) a_i) \\
& \Rightarrow S_{-1} \leq a_{t-1}\left(\frac{\alpha(r)}{\alpha(r)-1}\right)
\end{aligned}$$

Therefore

$$\sum_{i \in [t]} a_i \leq S_{-1} + a_t \leq \left(1 + \left(\frac{\alpha(r)}{\alpha(r)-1}\right)\right) a_t.$$

Setting $c(r) = \left(1 + \left(\frac{\alpha(r)}{\alpha(r)-1}\right)\right)$ we get the required result. $\alpha(2)$ is greater than $\frac{2}{\sqrt{3}}$ which implies that $c(2) < 9$. $\square$

Fact 1. *For every $c_1 \geq 0$, there exists c_2 s.t. $\sqrt{\sqrt{x}\log\frac{1}{x}} \leq c_1 + c_2 x$ where $0 \leq x \leq 1$.*

We will make use of the well-known Hoeffding inequality:

Theorem 5. *Let $X_1, \ldots, X_n$ be independent random variables such that X_i is strictly bounded within the interval $[a_i, b_i]$, then*

$$\mathbb{P}\left(\left|\sum_{i=1}^n X_i - \sum_{i=1}^n E[X_i]\right| \geq t\right) \leq 2e^{-\frac{2t^2}{\sum_{i=1}^n (b_i - a_i)^2}}.$$

2.2 Spectral Graph Theory Basics

Expander graphs are often seen as graphs which are close to random graphs. This idea is quantified by the following well-known fact known as the Expander Mixing Lemma which bounds the deviation between the number of edges between two subsets and the expected number in a random graph.

Theorem 6 (Expander-Mixing Lemma [LW03]). *For non-bipartite graph with maximum non-trivial eigenvalue λ,*

$$(\forall S, T \subseteq V)\ |E(S,T) - \frac{d|S||T|}{n}| \le \lambda\sqrt{|S||T|}$$

We can also get an analogue for bipartite graphs from the proof of the Expander Mixing Lemma. The following theorem states the general bound.

Theorem 7. *For a graph with maximum non-trivial eigenvalue λ,*

$$(\forall S, T \subseteq V)\ E(S,T) \le 2\frac{d|S||T|}{n} + \lambda\sqrt{|S||T|}.$$

We need the following theorem showing that expanders have small diameter in order to show no-expansion of large abelian lifts.

Theorem 8. *[Chu89] The diameter of a d-regular graph G with n vertices is upper bounded by* $\log(n)/\log(d/\lambda)$.

2.3 Lifts

In this section we formally define lifts of graphs and state some of their properties.

Definition 2 ($(\Gamma, S, \cdot)$-lift). Let Γ be a group, S be a set of size k and $\cdot$ be a faithful group action of Γ on S. A $(\Gamma, S, \cdot)$-lift of an n-vertex base graph $G(V,E)$ is a graph $H = (V \times S, E')$ obtained as follows: it has $k \times n$ vertices, where each vertex u of G is replaced by k vertices $\{u\} \times S$. For each edge (u,v) of G, we choose an element $g_{u,v} \in \Gamma$ and replace that edge by a perfect matching between $\{u\} \times S$ and $\{v\} \times S$ that is given by the edges (u_i, v_j) for which $g_{u,v} \cdot i = j$. We denote $|S| = k$ to be the order of the lift.

We note that if $S = \Gamma$ and the group action $\cdot$ is the left group operation itself, then $(\Gamma, S, \cdot)$-lifts are just Γ-lifts.

Remark 5 (Group Elements as Permutations). A faithful action of a group Γ on a set S induces an embedding from Γ to $Sym(S)$, where $Sym(S)$ is the symmetric group of S (group of all permutations of S). Thus, we can identify group elements with permutations of $|S| = k$ objects. Using this language, the set of edges of the lift H can be rewritten as $E' = \{(u_i, v_j) | (u,v) \in E, \pi_{u,v}(i) = j\}$, where $\pi_{u,v}$ is the permutation corresponding to the group element we have chosen for edge (u,v).

Besides Γ-lifts another interesting case of $(\Gamma, S, \cdot)$-lifts is when $\Gamma = Sym([k])$ (the symmetric group on k elements), $S = [k]$ and the group action $\cdot : \Gamma \times S \to S$ is defined by $\sigma \cdot t = \sigma(t)$, i.e., the action of the permutation on the corresponding element. Such lifts are known as *general lifts* or simply *k-lifts*. Recall that shift k-lifts are Γ-lifts where the group Γ is a cyclic group. We will use the term *abelian lifts* to refer to Γ-lifts where the group Γ is an abelian group.

Some initial easy observations can be made about the structure of any lift: (i) the lifted graph is also regular with the same degree as the base graph and (ii) the eigenvalues of A are also eigenvalues of A_H. Therefore we call the n eigenvalues of A the *old* eigenvalues and $n(k-1)$ other eigenvalues of A_H the *new* eigenvalues. We will denote by λ_{new} the largest new eigenvalue of H in magnitude, which we also refer to as the "first" new eigenvalue for simplicity.

Definition 3 (Generalized Signing). Given a base graph $G(V,E)$, a group Γ, a set S and an action $\cdot$ of Γ on S as in the above definition, we define a generalized signing of $G(V,E)$ as a function $s : E(G) \to \Gamma$. We use the convention that $s(u,v) = g$ then $s(v,u) = g^{-1}$. There is a bijection between signings and $(\Gamma, S, \cdot)$-lifts.

2.4 Spectrum of Lifts via Representation Theory

In this section, we characterize the spectrum of Γ-lifts as a union of the spectrum of certain matrices. We begin with some elementary facts on the representation theory of finite groups (see also [Art98, Ser97]).

Definition 4 (Representation)**.** A representation of a finite group Γ on a finite-dimensional vector space $\mathcal{V}$ is a homomorphism $\rho : \Gamma \to GL(\mathcal{V})$, where $GL(\mathcal{V})$ is the general linear group of $\mathcal{V}$. If the dimension of $\mathcal{V}$ is Δ, then we define the dimension of ρ to be Δ.

A trivial representation is one where $\mathcal{V} = \mathbb{C}$ and $\rho(g) = 1$ for all $g \in \Gamma$. A *permutation representation* is one where the matrices $\rho(g)$ correspond to permutation matrices. We next consider an interesting special case of permutation representations.

Definition 5 (Regular Representation)**.** For a group element $g \in \Gamma$, let e_g be the $|\Gamma|$-dimensional indicator vector of g and let $\mathbb{C}^\Gamma$ denote the vector space defined by the basis vectors $\{e_g\}_{g\in\Gamma}$. Let P_g denote the permutation matrix associated with the left action of g on Γ. Then $\rho(g) = P_g$ is a representation of Γ on $\mathcal{V} = \mathbb{C}^\Gamma$. This is known as the (left) regular representation of Γ on $\mathbb{C}^\Gamma$.

Definition 6 (Irreducible Representation)**.** For a representation $\rho : \Gamma \to GL(\mathcal{V})$, a subspace $\mathcal{W} \subset \mathcal{V}$ is invariant under ρ if $\rho(g)\mathcal{W} \subset \mathcal{W}$ for all $g \in \Gamma$. The representation ρ is irreducible (hereafter called irrep) if it has no (proper) invariant subspace.

A well-known theorem of Maschke shows that every permutation representation can be decomposed into a finite number of irreps. Our next theorem is a consequence of this result as applied to the regular representation.

Theorem 9 (Decomposition into irreps for Regular Representation [Ser97])**.** *Let ρ be the regular representation of Γ on $\mathbb{C}^\Gamma$. Then there exists a unitary matrix $U \in \mathbb{C}^{\Gamma\times\Gamma}$, an orthogonal decomposition $\mathbb{C}^\Gamma = \oplus\mathcal{V}_i$ and irreps $\rho_i : \Gamma \to GL(\mathcal{V}_i)$ such that $U\rho(g)U^{-1} = \oplus_i\rho_i(g)$ for every $g \in \Gamma$. Moreover, the invariant subspaces $\mathcal{V}_i$ are unique and the trivial representation is always one of the irreps.*

We next state a few properties of the irreps arising in Theorem 9 for abelian groups and cyclic groups.

Fact 2. *For abelian groups, the irreps in Theorem 9 are one-dimensional. In particular, for a cyclic group $\Gamma = \{c, c^2, \ldots, c^k\}$, the irreps are given by $\rho_1, \ldots, \rho_k : \Gamma \to GL(\mathbb{C})$, where $\rho_i(c^j) = \omega_i^j$, where ω_i is a primitive k-th root of unity.*

We note that when $k = 2$, the two roots of unity are $\omega_1 = 1$ and $\omega_2 = -1$, and the only non-trivial irrep is ρ_2, where $\rho_2(0) = 1, \rho_2(1) = -1$.

We now characterize the eigenvalues of Γ-lifts. We observe that the adjacency matrix of a Γ-lift is a $nk \times nk$ symmetric matrix, which has $n \times n$ blocks $B_{u,v}$, each of size $k \times k$; the block $B_{u,v}$ is the zero $k \times k$ matrix if (u, v) is not an edge in G; for every edge (u, v) of G, we have $B_{u,v} = P_{u,v}$, which is the permutation representation of the element $g = s(u, v) \in \Gamma$. The following theorem characterizes the spectrum of the lift in terms of the spectrum of certain smaller matrices. We note that even though G is an undirected graph, for the purposes of the theorem, we view it as a directed graph where if $(u, v) \in E$ then $(v, u) \in E$. Recall that when $s(u, v) = g$, then $s(v, u) = g^{-1}$.

Theorem 10. *[MS95] For $g \in \Gamma$, let G_g be the induced subgraph of G consisting of (directed) edges $(u, v) \in E$ such that $s(u, v) = g$, and let A_g be its adjacency matrix. The adjacency matrix of the lifted graph H is equal to $A_H = \sum_{g\in\Gamma} A_g \otimes P_g = U\big(\oplus_i \sum_{g\in\Gamma} A_g \otimes \rho_i(g)\big)U^{-1}$, for some unitary matrix U. Here ρ_i are the irreps of the regular (left) representation of Γ given in Theorem 9.*

The above theorem shows that there is some basis given by the columns of the matrix U such that A_H is block-diagonal in that basis, with blocks $D_i = \sum_{g\in\Gamma} A_g \otimes \rho_i(g)$. In particular, the spectrum of H is equal to the spectrum of the set of matrices D_i. We note that since for any group ρ_1 is the trivial, one-dimensional representation, it follows that $D_1 = A_G$, the adjacency matrix of the original graph. This is consistent with the observation in Section 2.3 that all the "old" eigenvalues of G are also eigenvalues of H.

We now specialize Theorem 10 to the case of cyclic groups to characterize the spectrum of shift k-lifts. For a shift k-lift of a graph $G = (V, E)$ with adjacency matrix A, which is given by the signing

$(s(i,j) = g_{i,j})_{(i,j)\in E}$, define the following family of Hermitian matrices $A_s(\omega)$ parameterized by ω where ω is a k-th primitive root of unity.

$$[A_s(\omega)]_{ij} = \begin{cases} 0, & \text{if } A_{ij} = 0 \\ \omega^{g_{i,j}}, & \text{if } A_{ij} = 1 \end{cases}$$

The following corollary regarding $A_s(t)$ follows from Theorem 10 and Fact 2.

Corollary 1. Let $G = (V, E)$ be a graph and H be a shift k-lift of G with the corresponding signing of the edges $(s(i,j) = g_{i,j})_{(i,j)\in E}$, where $g_{i,j} \in C_k$. Then the set of eigenvalues of H are given by

$$\bigcup_{\omega:\ \omega \text{ is a } k\text{-th primitive root of unity}} \text{eigenvalues}\left(A_s(\omega)\right).$$

The above simplifies significantly for 2-lifts as noted in the next corollary.

Corollary 2. When $k = 2$, the set of eigenvalues of a 2-lift H is given by the eigenvalues of A and the eigenvalues of A_s, where A_s is the signed adjacency matrix corresponding to the signing s, with entries from $\{0, 1, -1\}$.

3 No-expansion of Abelian Lifts

In this section we show that it is impossible to find (even slightly) expanding graphs which are lifts by large abelian groups Γ. By Theorem 8, we know that if a graph is an expander, then it has small diameter. We show that if the size of the (abelian) group Γ is large, then *all* Γ-lifts of any base graph have large diameter, and hence they cannot be expanders. We prove Theorem 3.

Theorem 3. *For every n-vertex d-regular graph G, $\epsilon \in (0,1)$, and abelian group Γ of size at least*

$$k = exp\left(\frac{nd \log \frac{1}{\epsilon} + \log n}{\log \frac{1}{e\epsilon}}\right),$$

all Γ-lifts of G have second largest eigenvalue at least ϵd. In particular, when $k = 2^{\Omega(nd)}$, there is no Γ-lift H of any n-vertex d-regular graph G with $\lambda(H) = \mathcal{O}(\sqrt{d})$ whenever Γ is an abelian group of order k.

Proof. Let Γ be an abelian group of order k and $G = (V, E)$ be a base graph on n-vertices that is d-regular. Let $e_1, \ldots, e_{nd/2}$ be an arbitrarily chosen ordering of the edges E. Let H be a lift graph obtained using a Γ-lift. Recall that the signing of the edges of the base graph correspond to group elements, which in turn correspond to permutations of k elements. Let these signing of the edges be $(\sigma_e)_{e\in E(G)}$. For notational convenience, let us define a layer L_i of H to be the set of vertices $\{v_i : v \in V\}$. We note that H has k layers.

Let us fix an arbitrary vertex v in G. Let Δ denote the diameter of H. This implies that for every $j = 2, \ldots, k$ there exists a path of length at most Δ in H from v_1 to a vertex in L_j. A layer j is reachable within distance Δ in H iff there exists a walk $e_1, e_2, \ldots, e_t$ from v of length $t \le \Delta$ in G such that $\sigma_{e_t}\sigma_{e_{t-1}} \ldots \sigma_{e_2}\sigma_{e_1}(1) = j$. Thus the set of layers reachable within distance Δ in H is contained in the set $S = \{\sigma_{e_t} \ldots \sigma_{e_1}(1) : e_1, \ldots, e_t \text{ is a walk from } v \text{ in } G \text{ of length } t \le \Delta\}$. Since the group Γ is abelian, $S \subseteq \{\sigma_{e_1}^{a_1}\sigma_{e_2}^{a_2} \ldots \sigma_{e_{nd/2}}^{a_{nd/2}}(1) \mid \sum_{i=1}^{nd/2} |a_i| \le \Delta\} =: T$. Since H has k layers, the cardinality of S is at least k.

The number of integral a_i's satisfying $\sum_{i=1}^{nd/2} |a_i| \le \Delta$ is at most $\binom{(nd/2)+\Delta}{(nd/2)} \cdot 2^{(nd/2)}$. Therefore,

$$k \le |T| \le \binom{\frac{nd}{2} + \Delta}{\frac{nd}{2}} 2^{\frac{nd}{2}} \le \left(2e\left(1 + \frac{2\Delta}{nd}\right)\right)^{\frac{nd}{2}} \le (2e)^{\frac{nd}{2}} e^{\Delta}.$$

If $\lambda_2(G) > \epsilon d$, then it follows that $\lambda_2(H) > \epsilon d$ for every Γ and the result follows. So, we may assume that $\lambda_2(G) \le \epsilon d$. Since H has nk vertices, using Theorem 8, we have $\Delta \le (\log nk)/\log(d/\lambda_2)$. Thus, if

$\lambda_{new} \le \epsilon d$, then $\Delta \le (\log nk)/\log(1/\epsilon)$ and consequently,

$$k \le (2e)^{\frac{nd}{2}} e^{\frac{\log nk}{\log \frac{1}{\epsilon}}}.$$

Rearranging the terms, we obtain that

$$k \le (2e)^{\overline{2\left(1-\frac{1}{\log \frac{1}{\epsilon}}\right)}^{\,nd}} exp\left(\frac{\log n}{\left(\log \frac{1}{\epsilon}\right)\left(1-\frac{1}{\log\frac{1}{\epsilon}}\right)}\right) \le exp\left(\frac{nd\log\frac{1}{\epsilon}+\log n}{\log\frac{1}{e\epsilon}}\right).$$

□

4 Expansion of Random 2-lifts

4.1 Overview

In this section, we sketch a proof of Theorem 11 that is a slightly weaker version of Theorem 1 (weaker by a multiplicative factor of four). The proof of this weaker result captures the main ideas involved in the proof of Theorems 1 and 2. Theorem 1 follows from Theorem 2 as a special case. We present the full proof of Theorem 2 in Section 6.

Theorem 11. *Let G be a d-regular n-vertex graph with non-trivial eigenvalues at most λ in absolute value where $\sqrt{d} \le \lambda, 2 \le d \le \sqrt{\frac{n}{3\ln n}}$, and H be a uniformly random 2-lift of G. Let λ_{new} be the largest new eigenvalue of H in magnitude. Then, there exists a constant c such that*

$$\lambda_{new} \le 4\lambda + c\max\left(\sqrt{\lambda \log d}, \sqrt{d}\right)$$

with probability at least $1 - e^{-n/d^2}$.

We note that G is moderately expanding such that $\lambda \le \frac{d}{\log d}$, then we get $\lambda_{new} = 4\lambda + \mathcal{O}(\sqrt{d})$. To prove this theorem, we require the following concentration inequality.

(Karthik: We are assuming that $\lambda \ge \sqrt{d}$. We need to remove this from the theorem statement (and lemma statements) and add a justification.) ??

Lemma 3. *Let G be a d-regular graph with non-trivial eigenvalues at most λ in absolute value where $\sqrt{d} \le \lambda, 2 \le d \le \sqrt{\frac{n}{3\ln n}}$. Let H be a uniformly random 2-lift of G, with corresponding signed adjacency matrix A_s. The following statements hold with probability at least $1 - e^{-n/d^2}$ over the choice of the random signing:*

1. *For all $u_1, \ldots, u_r \in \{0, \pm 1\}^n$, and $v_1, \ldots, v_\ell \in \{0, \pm 1\}^n$ satisfying*
   - *(I) $S(u_i) \cap S(u_j) = \emptyset$ for every $i, j \in [r]$ and $S(v_i) \cap S(v_j) = \emptyset$ for every $i, j \in [\ell]$, and*
   - *(II) Either $|S(u_i)| > n/d^2$ for every $i \in [r]$ with non-zero u_i, or $|S(v_i)| > n/d^2$ for every $i \in [\ell]$ with non-zero v_i,*

   we have

$$\left|\sum_{i\le j}(2^{-i}u_i^T)A_s(2^{-j}v_j)\right| \le 377\max(\sqrt{\lambda\log d}, \sqrt{d})\sum_{i=1}^{r}|S(u_i)|2^{-2i} + \left(\frac{\lambda}{5} + 10^{12}\sqrt{d}\right)\sum_{j=1}^{\ell}|S(v_j)|2^{-2j}.$$

2. *For all $u_1, \ldots, u_r \in \{0, \pm 1\}^n$, and $v_1, \ldots, v_\ell \in \{0, \pm 1\}^n$ satisfying (I), (II) and*

   (III) $|S(u_i)| > |S(v_j)|$ for every $i \in [r], j \in [\ell]$ with non-zero u_i,

we have

$$\left|\sum_{i\le j}(2^{-i}u_i^T)A_s(2^{-j}v_j)\right| = 31\max\left(\sqrt{\lambda\log d},\sqrt{d}\right)\left(\sum_{i=1}^{r}|S(u_i)|2^{-2i}+\sum_{j=1}^{\ell}|S(v_j)|2^{-2i}\right).$$

We will now prove Theorem 11 using the lemma above.

Proof of Theorem 11. The first new eigenvalue of the lift is $\lambda_{new} = \max_{x\in\mathbb{R}^n}|x^TA_sx/x^Tx|$. To prove an upper bound on λ_{new}, we will bound $|x^TA_sx/x^Tx|$ for all x with high probabliity. In particular, assuming that the concentration inequalities given by Lemma 3 holds, we will show that

$$\left|x^TA_sx\right| \le 4\left(\lambda+2\cdot 10^{13}\sqrt{d}\right)\|x\|^2.$$

By re-scaling we may assume that the maximum entry of x is less than $1/2$ in absolute value. Next, we use Lemma 1 to find a vector $y\in\{\pm1/2,\pm1/4,\ldots\}^n$ such that $|x^TA_sx|\le|y^TA_sy|$ and $\|y\|^2\le 4\|x\|^2$. We will prove a bound on $|y^TA_sy|$, which in turn will imply the desired bound on $|x^TA_sx|$. Let us consider the diadic decomposition of $y=\sum_{i=1}^{\infty}2^{-i}u_i$ obtained as follows: a coordinate of u_i is 1 if the corresponding coordinate of y is 2^{-i}, it is -1 if the corresponding coordinate in y is -2^{-i}, and is zero otherwise. We note that $S(u_i)\cap S(u_j)=\emptyset$ for every pair $i,j\in\mathbb{N}$.

Next, we partition the set of vectors u_i's based on their support sizes. Let $M := \{i\in\mathbb{N} : |S(u_i)|\le n/d^2\}$ and $L := \{i\in\mathbb{N} : |S(u_i)| > n/d^2\}$ (M and L for mini and large supports respectively). Correspondingly, define $y_M := \sum_{i\in M}2^{-i}u_i$ and $y_L=\sum_{i\in L}2^{-i}u_i$. We note that $y=y_M+y_L$, $\|y\|^2=\|y_M\|^2+\|y_L\|^2=\sum_{i\in\mathbb{N}}|S(u_i)|2^{-2i}$, and

$$|y^TA_sy|\le|y_M^TA_sy_M|+2|y_M^TA_sy_L|+|y_L^TA_sy_L|.$$

We next bound each term in the following three claims.

Claim 1.

$$|y_M^TA_sy_M|\le\left(\lambda+\frac{4}{d}\right)\|y_M\|^2.$$

Proof. Let y'_M be a vector obtained from y_M by taking the absolute values of the coordinates. Then $\|y_M\|^2=\|y'_M\|^2$ and $|y_M^TA_sy_M|\le y_M'^TAy'_M$. Let J be the $n\times n$ matrix with all entries being 1. We have

$$y_M'^TAy'_M = y_M'^T\left(A-\frac{d}{n}J\right)y'_M + y_M'^T\left(\frac{d}{n}J\right)y'_M \le \lambda\|y'_M\|^2 + y_M'^T\left(\frac{d}{n}J\right)y'_M.$$

Above, we have used the fact that $A-\frac{d}{n}J$ has the same set of eigenvalues as A except for the first eigenvalue which was d for the matrix A and is now zero. It remains to bound $y_M'^T\left(\frac{d}{n}J\right)y'_M$. Consider the diadic decomposition of $y'_M=\sum_{i\in M}2^{-i}u'_i$, where the coordinates of u'_i are the absolute values of the coordinates of u_i.

$$\begin{aligned}
y_M'^T\left(\frac{d}{n}J\right)y'_M &\le 2\sum_{i\in M}\sum_{j\in M:j\ge i}\frac{d}{n}2^{-i}|S(u_i)|2^{-j}|S(u_j)| \\
&\le 2\sum_{i\in M}\frac{1}{d}2^{-2i}|S(u_i)|\sum_{j\in M:j\ge i}2^{i-j} \qquad (\text{since } |S(u_j)|\le n/d^2\ \forall\ j\in M) \\
&\le \frac{4}{d}\|y'_M\|^2.
\end{aligned}$$

□

Claim 2.

$$|y_L^TA_sy_L|\le\left(\frac{2\lambda}{5}+(2\cdot10^{12}+754)\max\left(\sqrt{\lambda\log d},\sqrt{d}\right)\right)\|y_L\|^2.$$

Proof. By triangle inequality,

$$
\begin{aligned}
|y_L^T A_s y_L| &= \left| \sum_{i,j\in L} (2^{-i}u_i^T) A_s (2^{-j}u_j) \right| \\
&\le \left| \sum_{i,j\in L: i\le j} (2^{-i}u_i) A_s (2^{-j}u_j) \right| + \left| \sum_{i,j\in L: i> j} (2^{-i}u_i) A_s (2^{-j}u_j) \right|.
\end{aligned}
$$

We bound each term using the first part of Lemma 3. For both terms, our choice is $r \leftarrow \min\{i \in L\}$, $\ell = r$, $u_i \leftarrow u_i$ if $i \in L$ and $u_i \leftarrow \overline{0}$ if $i \notin L$, $v_i = u_i$ for every $i \in [r]$, where $\overline{0}$ is the all-zeroes vector. We note that the conditions (i) and (ii) of the lemma are satisfied by this choice since every pair $S(u_i), S(u_j)$ is mutually disjoint and $|S(u_i)| > \frac{n}{d^2}$ for all $i \in L$. Consequently,

$$
\begin{aligned}
|y_L^T A_s y_L| &\le 754 \max\left(\sqrt{\lambda \log d}, \sqrt{d}\right) \sum_{i\in L} |S(u_i)| 2^{-2i} + \left(\frac{\lambda}{5} + 2\cdot 10^{12}\sqrt{d}\right) \sum_{j\in L} |S(u_j)| 2^{-2j} \\
&\le \left(\frac{2\lambda}{5} + (2\cdot 10^{12} + 754) \max\left(\sqrt{\lambda \log d}, \sqrt{d}\right)\right) \|y_L\|^2.
\end{aligned}
$$

$\square$

Claim 3.

$$
|y_M^T A_s y_L| \le 408 \max\left(\sqrt{\lambda \log d}, \sqrt{d}\right) \|y_M\|^2 + \left(\frac{\lambda}{5} + (10^{12} + 31) \max\left(\sqrt{\lambda \log d}, \sqrt{d}\right)\right) \|y_L\|^2.
$$

Proof. By triangle inequality,

$$
\begin{aligned}
|y_M^T A_s y_L| &= \left| \sum_{i\in M, j\in L} (2^{-i}u_i^T) A_s (2^{-j}u_j) \right| \\
&\le \left| \sum_{i\in M, j\in L: i\le j} (2^{-i}u_i) A_s (2^{-j}u_j) \right| + \left| \sum_{i\in M, j\in L: i> j} (2^{-i}u_i) A_s (2^{-j}u_j) \right|.
\end{aligned}
$$

We bound the first and second terms by the first and second parts of Lemma 3 respectively. Let $\overline{0}$ be the all-zeroes vector. For the first term, our choice is $r \leftarrow \min\{i \in M\}$, $\ell \leftarrow \min\{i \in L\}$, $u_i \leftarrow u_i$ if $i \in M$ and $u_i \leftarrow \overline{0}$ if $i \notin M$, and $v_i \leftarrow u_i$ if $i \in L$ and $v_i \leftarrow \overline{0}$ if $i \notin L$. For the second term, our choice is $r \leftarrow \min\{i \in L\}$, $\ell \leftarrow \min\{i \in M\}$, $u_i \leftarrow u_i$ if $i \in L$ and $u_i \leftarrow \overline{0}$ if $i \notin L$, and $v_i \leftarrow u_i$ if $i \in M$ and $v_i \leftarrow \overline{0}$ if $i \notin M$. The conditions (i), (ii) and (iii) of the lemma are satisfied for the respective choices since every pair $S(u_i), S(u_j)$ is mutually disjoint, $|S(u_i)| > \frac{n}{d^2}$ for all $i \in L$ and $|S(u_i)| > n/d^2 \ge |S(u_j)|$ for every $i \in L, j \in M$. Consequently,

$$
\begin{aligned}
|y_M^T A_s y_L| &\le 377 \max\left(\sqrt{\lambda \log d}, \sqrt{d}\right) \sum_{i\in M} |S(u_i)| 2^{-2i} + \left(\frac{\lambda}{5} + 10^{12}\sqrt{d}\right) \sum_{j\in L} |S(u_j)| 2^{-2j} \\
&\quad + 31 \max\left(\sqrt{\lambda \log d}, \sqrt{d}\right) \left(\sum_{j\in L} |S(u_j)| 2^{-2j} + \sum_{j\in M} |S(u_j)| 2^{-2j}\right) \\
&\le 408 \max\left(\sqrt{\lambda \log d}, \sqrt{d}\right) \|y_M\|^2 + \left(\frac{\lambda}{5} + (10^{12} + 31) \max\left(\sqrt{\lambda \log d}, \sqrt{d}\right)\right) \|y_L\|^2.
\end{aligned}
$$

$\square$

From the above three claims, we have

$$\begin{aligned}|y^T A_s y| &\leq \left(\lambda + 817 \max\left(\sqrt{\lambda \log d}, \sqrt{d}\right)\right) \|y_M\|^2 + \left(\frac{4\lambda}{5} + (4 \cdot 10^{12} + 816) \max\left(\sqrt{\lambda \log d}, \sqrt{d}\right)\right) \|y_L\|^2 \\ &\leq \lambda + 5 \cdot 10^{12} \max\left(\sqrt{\lambda \log d}, \sqrt{d}\right) \|y\|^2.\end{aligned}$$

Therefore, we have

$$\begin{aligned}|x^T A_s x| &\leq |y^T A_s y| \\ &\leq \left(\lambda + 5 \cdot 10^{12} \max\left(\sqrt{\lambda \log d}, \sqrt{d}\right)\right) \|y\|^2 \\ &\leq 4\left(\lambda + 5 \cdot 10^{12} \max\left(\sqrt{\lambda \log d}, \sqrt{d}\right)\right) \|x\|^2.\end{aligned}$$

□

We note that in the above proof, the multiplicative factor of 4 is a by-product of the discretization of x. This can be avoided if we do not discretize x straightaway, but instead "push" the discretization a little deeper into the proof. Indeed, we can see that the proof of Claim 1 where we bound $|y_M^T(A - \frac{d}{n}J)y_M|$ by $\lambda\|y_M\|^2$ does not require y_M to be a discretized vector. This is how we are able to prevent the multiplicative factor loss to obtain Theorem 1.

5 Concentration Inequality

In order to prove Lemma 3 we need to upper bound the sum $|\sum_{i \leq j} 2^{-i-j} u_i^T A_s v_j|$ for all sets of vectors $\{u_1, \ldots, u_r\}$, $\{v_1, \ldots, v_\ell\}$ satisfying the assumptions of the lemma. To begin with, one could try to use the triangle inequality and upper bound each term $|u_i^T A_s v_j|$ separately for each i, j. We note that $u_i^T A_s v_j$ is a sum of $|E(S(u_i), S(v_j))|$ iid random variables with mean zero (one for each edge between $S(u_i)$ and $S(v_j)$). By the expander mixing lemma (Theorem 7), we may upper bound the size of $E(S(u_i), S(v_j))$ by $2d|S(u_i)||S(v_j)|/n + \lambda\sqrt{|S(u_i)||S(v_j)|}$. Depending on which of these two terms in the RHS dominates, we have two cases. For each case, we use a different concentration bound (Lemma 4 and Corollary 3). We begin with the needed concentration bounds.

5.1 Concentration Bounds

Lemma 4. *Let G be a d-regular, n-vertex graph with non-trivial eigenvalues at most λ in absolute value where $2 \leq d \leq \left(\frac{2n}{3 \ln n}\right)^2$ and $\lambda \geq \sqrt{d}$. Let H be a uniformly random 2-lift of G, with corresponding signed adjacency matrix A_s. The following property holds with probability atleast $1 - e^{-(n \log d)/\sqrt{d}}$ (over the random choice of signings):*

For every $r \in \{0, 1, \ldots, 1/2 \log d\}$, every $a, b_0, b_1, \ldots, b_r \in \{0, \pm 1\}^n$ satisfying

(i) $S(b_i) \cap S(b_j) = \emptyset \ \forall\ i, j \in [r], i \neq j$,

(ii) $|S(a)| \geq 2^{2i}|S(b_i)| \ \forall\ i \in [r]$, *and*

(iii) $\frac{d}{\lambda}\sqrt{|S(b_i)||S(a)|} \geq n \ \forall\ i \in [r]$ *with non-zero* b_i,

we have

$$\left|a^T A_s \left(\sum_{i=0}^{r} 2^i b_i\right)\right| \leq 14\sqrt{\frac{d}{n}|S(a)|^2 \left(\sum_{i=0}^{r} |S(b_i)| 2^{2i}\right) \log\left(\frac{2n}{|S(a)|}\right)}.$$

Proof. For notational convenience, let $b = \sum_{i=0}^{r} 2^i b_i$. Fix $a, b_1, b_2, \ldots, b_r \in \{0, \pm 1\}^n$. Then $a^T A_s b$ is a sum of independent random variables with mean 0 one for each edge between $S(a)$ and $S(b_i)$. This is because the intersection between the support of any two vectors b_i and b_j is empty. The sum of squares of the difference between the maximum and the minimum values of these variables is at most

$\sum_{i=1}^{r} 4E(S(b_i), S(a))2^{2i}$. For vectors $a, b_1, \ldots, b_r$ satisfying (i) and (ii), by the Expander Mixing Lemma, we have $E(S(b_i), S(a)) \leq 3\frac{d|S(b_i)||S(a)|}{n}$. We note that this inequality holds even if b_i is a zero vector.

By Theorem 5,

$$Pr\left(|a^T A_s b| > 14\sqrt{\frac{d}{n}|S(a)|^2\left(\sum_i |S(b_i)|2^{2i}\right)\log\left(\frac{2n}{|S(a)|}\right)}\right) \leq 2exp\left(-\frac{98|S(a)|}{3}\log\left(\frac{2n}{|S(a)|}\right)\right)$$

Now fixing the values of the support sizes $\alpha = |S(a)|, \beta_i = |S(b_i)|$, the number of possible choices for a is at most $\binom{n}{\alpha} * 2^{\alpha} \leq exp\left(3\alpha\log(\frac{2n}{\alpha})\right)$. Similarly the number of possible choices for each b_i is atmost $\exp\left(3\beta_i\log(\frac{2n}{\beta_i})\right)$. Therefore the total number of choices for b is at most $exp\left(\sum_{i=1}^{r} 3\beta_i\log(\frac{2n}{\beta_i})\right)$. Since each $\alpha, \beta_i \leq n$, we can replace each β_i by its upper bound $\alpha 2^{-2i}$. Hence, using Lemma 2,

$$\exp\left(\sum_{i=1}^{r} 3\beta_i\log\left(\frac{2n}{\beta_i}\right)\right) \leq exp\left(3\sum_{i=1}^{r}\alpha 2^{-2i}\log(\frac{2n}{\alpha 2^{-2i}})\right) \leq exp\left(27\alpha\log\left(\frac{2n}{\alpha}\right)\right).$$

Therefore, the total number of choices of $a, b_1, \ldots, b_r$ of sizes $\alpha, \beta_1, \ldots \beta_r$ respectively is at most

$$\exp\left(30\alpha\log\left(\frac{2n}{\alpha}\right)\right).$$

By taking a union bound over the choices of vectors with the fixed support sizes, the probability of the existence of a set of vectors $a, b_1, \ldots, b_r$ with sizes $\alpha, \beta_1, \ldots, \beta_r$ respectively and satisfying (i) and (ii) is bounded by

$$2exp\left(-\frac{8\alpha}{3}\log\left(\frac{2n}{\alpha}\right)\right) \leq 2exp\left(-\frac{8n}{3\sqrt{d}}\log\left(2\sqrt{d}\right)\right).$$

Above, we have used that $\alpha \geq n/\sqrt{d}$ which follows since $\alpha = |S(a)| \geq n\lambda/d \geq n/\sqrt{d}$ by (ii) and (iii). Next, let us bound the number of choices for the support sizes of the vectors $a, b_1, \ldots, b_r$. The number of choices for the support sizes is at most $n^{2+(1/2)\log d}$. Therefore taking the union bound over the choice of the support sizes, we get that the total probability is at most

$$2exp\left((2 + (1/2)\log d)\ln n\right) exp\left(-\frac{8n}{3\sqrt{d}}\log(2\sqrt{d})\right) \leq exp\left(-\frac{n\log d}{\sqrt{d}}\right).$$

$\square$

Lemma 5. *Let G be a d-regular, n-vertex graph with non-trivial eigenvalues at most λ in absolute value, where $2 \leq d \leq \sqrt{\frac{n}{3\ln n}}$, and H be a uniformly random 2-lift of G, with corresponding signed adjacency matrix A_s. The following property holds with probability at least $1 - e^{-3n/d^2}$ (over the random choice of signings):*

For every $a, b \in \{0, \pm 1\}^n, q, w \in \{1, \ldots, n\}$ satisfying

(i) $|S(a)| \leq q$, $|S(b)| \leq w$, $S(b) \subset N_G(S(a))$,

(ii) $q \leq w \leq dq$,

(iii) $w > \frac{n}{d^2}$, and

(iv) $\frac{d}{\lambda}\sqrt{qw} < n$,

we have

$$|a^T A_s b| \leq 10\sqrt{\lambda\sqrt{q}w^{3/2}\log\left(\frac{2dq}{w}\right)}. \tag{1}$$

Here, $N_G(S(a))$ denotes the set of neighbors of $S(a)$ formally defined as $\{v \mid \exists u \in S(a) \text{ with } (u,v) \in E\}$.

Proof. For a pair of vectors $a, b \in \{0, \pm 1\}^n$ and $q, w \in \{1, \ldots, n\}$, let $Bad(a, b, q, w)$ denote the event that inequality (1) is violated. We need to upper bound the probabillity that there exists (a, b, q, w) satisfying (i), (ii), (iii) and (iv) such that $Bad(a, b, q, w)$ happens. We note that the sum $a^T A_s b$ over random choices of A_s is a sum of independent random variables chosen from $\{\pm 2, \pm 1\}$, all of which have mean 0. The number of such random variables being summed is at most $E(S(a), S(b))$, i.e. the number of edges between $S(a)$ and $S(b)$.

Therefore for a fixed a, b, q, w by applying the Hoeffding inequality (Theorem 5), we get that

$$P(Bad(a,b,q,w)) \leq 2exp\left(-\frac{50\lambda\sqrt{q}w^{3/2}\log\left(\frac{2dq}{w}\right)}{E(S(a),S(b))}\right).$$

Now using (iv) and the expander mixing lemma (Theorem 7), we have

$$E(S(a), S(b)) \leq 2d|S(a)||S(b)|/n + \lambda\sqrt{|S(a)||S(b)|} \leq 2dqw/n + \lambda\sqrt{qw} \leq 3\lambda\sqrt{qw}.$$

Substituting this in the previous expression, we obtain

$$P(Bad(a,b,q,w)) \leq 2exp\left(-(50/3)w\log\left(\frac{2dq}{w}\right)\right).$$

We will use the union bound now. For this purpose, we will first fix q, w and the size of the support of a and b. We take a union bound over all possible choices of a, b of that fixed size, and then take a union bound over all choices of the support sizes. For fixed support sizes $\alpha = |S(a)|, \beta = |S(b)|$, we observe that the total number of choices for the support sets for a are $\binom{n}{\alpha}$. Now, since $S(b)$ is a subset of $N_G(S(a))$, the number of choices of $S(b)$ is bounded by $\binom{d\alpha}{\beta}$. Also, since each entry in a, b is 0 or ± 1 the total number of choices for a and b is at most

$$\binom{n}{\alpha}2^\alpha\binom{d\alpha}{\beta}2^\beta \leq exp\left(3\alpha\log\left(\frac{2n}{\alpha}\right)\right) exp\left(3\beta\log\left(\frac{2d\alpha}{\beta}\right)\right)$$

We will first show upper bounds on each of these terms. Since $w \geq \frac{n}{d^2}$, by (ii), we have $q \geq \frac{n}{d^3}$. Also, $\alpha = |S(a)| \leq q, \beta = |S(b)| \leq w$. Therefore,

$$\begin{aligned}
exp\left(3\alpha\log\left(\frac{2n}{\alpha}\right)\right) &\leq exp\left(3q\log\left(\frac{2n}{q}\right)\right)\\
&\leq exp\left(9q\log(2d)\right)\\
&= exp\left(9\frac{\frac{q}{w}\log(2d)}{\log\left(2d\frac{q}{w}\right)}\cdot w\log\left(2d\frac{q}{w}\right)\right)\\
&\leq exp\left(9w\log\left(2d\frac{q}{w}\right)\right).
\end{aligned}$$

The last line follows from the fact that $x\log(d)/\log(2dx)$ is bounded by 1 for $x \in [1/d, 1]$ and that $\frac{q}{w} \in [1/d, 1]$. Further,

$$exp\left(3\beta\log\left(\frac{2d\alpha}{\beta}\right)\right) \leq exp\left(3\beta\log\left(\frac{2dq}{\beta}\right)\right) \leq exp\left(3w\log\left(\frac{2dq}{w}\right)\right).$$

The last inequality follows by the fact that $x\log\frac{2c}{x}$ is an increasing function if $x < c$. Therefore, by union bound we get that the probability of a bad event for fixed q, w and support sizes $\alpha = |S(a)|, \beta = |S(b)|$ is at most

$$2exp\left(-(14/3)w\log\frac{4dq}{w}\right) \leq 2exp\left(-\frac{14n}{3d^2}\log\frac{4dq}{w}\right) \leq 2exp\left(-\frac{14n}{3d^2}\log 2\right).$$

Now the number of choices of the supports is at most n^2, number of choices for q, w is at most n^2 and therefore,

$$P(\exists(a,b,q,w) \text{ satisfying } (i), (ii), (iii), \text{ and } (iv){:}Bad(a,b,q,w)) \leq 2n^4 exp\left(-\frac{14n}{3d^2}\log 2\right) \leq exp\left(-\frac{3n}{d^2}\right).$$

□

Corollary 3. Let G be a d-regular, n-vertex graph with non-trivial eigenvalues at most λ in absolute value, where $2 \leq d \leq \sqrt{\frac{n}{3\ln n}}$, and H be a uniformly random 2-lift of G, with corresponding signed adjacency matrix A_s. The following property holds with probability at least $1 - e^{-3n/d^2}$ (over the random choice of signings):

For every $a, b \in \{0, \pm 1\}^n$ satisfying

(i) $|S(a)| \leq |S(b)| \leq d|S(a)|$,

(ii) $|S(b)| > \frac{n}{d^2}$, and

(iii) $\frac{d}{\lambda}\sqrt{|S(a)||S(b)|} < n$,

we have

$$|a^T A_s b| \leq 10\sqrt{\lambda\sqrt{|S(a)||S(b)|}|S(b)|\log\left(\frac{2d|S(a)|}{|S(b)|}\right)}. \tag{2}$$

Proof. For every a, b, we apply the bound from Lemma 5 on $|a^T A_s b'|$ with $q = |S(a)|, w = |S(b)|$ where b' is the same as b restricted to the coordinates in $S(b) \cap N_G(S(a))$. We observe that $|a^T A_s b| = |a^T A_s b'|$ and hence the corollary.

□

5.2 Proof of Lemma 3

Next, we use Corollary 3 and Lemma 4 to prove Lemma 3. We restate the Lemma for the sake of presentation.

Lemma 3. *Let G be a d-regular graph with non-trivial eigenvalues at most λ in absolute value where $\sqrt{d} \leq \lambda, 2 \leq d \leq \sqrt{\frac{n}{3\ln n}}$. Let H be a uniformly random 2-lift of G, with corresponding signed adjacency matrix A_s. The following statements hold with probability at least $1 - e^{-n/d^2}$ over the choice of the random signing:*

1. *For all $u_1, \ldots, u_r \in \{0, \pm 1\}^n$, and $v_1, \ldots, v_\ell \in \{0, \pm 1\}^n$ satisfying*

   (I) $S(u_i) \cap S(u_j) = \emptyset$ for every $i, j \in [r]$ and $S(v_i) \cap S(v_j) = \emptyset$ for every $i, j \in [\ell]$, and

   (II) Either $|S(u_i)| > n/d^2$ for every $i \in [r]$ with non-zero u_i, or $|S(v_i)| > n/d^2$ for every $i \in [\ell]$ with non-zero v_i,

   we have

$$\left|\sum_{i \leq j}(2^{-i}u_i^T)A_s(2^{-j}v_j)\right| \leq 377\max(\sqrt{\lambda\log d}, \sqrt{d})\sum_{i=1}^{r}|S(u_i)|2^{-2i} + \left(\frac{\lambda}{5} + 10^{12}\sqrt{d}\right)\sum_{j=1}^{\ell}|S(v_j)|2^{-2j}.$$

2. *For all $u_1, \ldots, u_r \in \{0, \pm 1\}^n$, and $v_1, \ldots, v_\ell \in \{0, \pm 1\}^n$ satisfying (I), (II) and*

   (III) $|S(u_i)| > |S(v_j)|$ for every $i \in [r], j \in [\ell]$ with non-zero u_i,

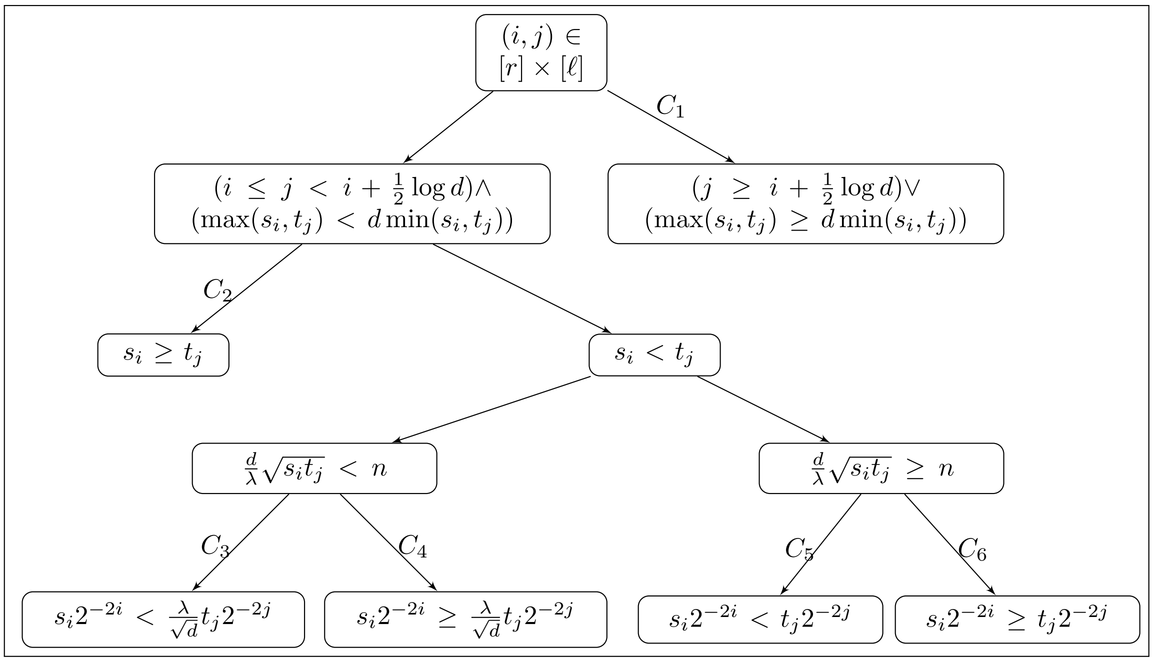
   we have

$$\left|\sum_{i \leq j}(2^{-i}u_i^T)A_s(2^{-j}v_j)\right| = 31\max\left(\sqrt{\lambda\log d}, \sqrt{d}\right)\left(\sum_{i=1}^{r}|S(u_i)|2^{-2i} + \sum_{j=1}^{\ell}|S(v_j)|2^{-2i}\right).$$

Proof. For notational convenience, we will replace $|S(u_i)|$ by s_i and $|S(v_j)|$ by t_j. We split the sum

$$\sum_{i \le j} (2^{-i} u_i^T) A_s (2^{-j} v_j)$$

into several subcases depending on i, j and the sizes of $S(u_i)$ and $S(v_j)$ and use the triangle inequality. Figure 5.2 summarizes the splitting of (i, j) into various terms depending on the various values of i, j, s_i and t_j. Next, we bound each of the terms separately. By Lemma 4 and Corollary 3, we know that A_s satisfies the property mentioned in both of them with probability atleast $1 - 2e^{-3n/d^2}$. We bound the terms assuming that A_s satisfies the property mentioned in Lemma 4 and Corollary 3.



Claim 4.

$$\left| \sum_{(i,j) \in C_1} (2^{-i} u_i^T) A_s (2^{-j} v_j) \right| \le 3\sqrt{d} \left(\sum_{i \in [r]} s_i 2^{-2i} + \sum_{j \in [\ell]} t_j 2^{-2j} \right).$$

Proof. The sum is conditioned over the set of tuples (i, j) in C_1, where

$$C_1 = \left\{ (i \in [r], j \in [\ell]) \mid (j \ge i + \frac{1}{2}\log d) \text{ or } (\max(s_i, t_j) \ge d\min(s_i, t_j)) \right\}.$$

By triangle inequality,

$$\left| \sum_{(i,j) \in C_1} 2^{-i-j} u_i^T A_s v_j \right| \le \left| \sum_{(i,j) \in [r] \times [\ell] : j \ge i + \frac{1}{2}\log d} 2^{-i-j} u_i^T A_s v_j \right| + \left| \sum_{\substack{(i,j) \in [r] \times [\ell] : i \le j < i + \frac{1}{2}\log d, \\ \max(s_i, t_j) \ge d\min(s_i, t_j)}} 2^{-i-j} u_i^T A_s v_j \right|$$

We note that the number of edges out of any set S is bounded by $d|S|$. So, $|u_i^T A_s v_j| \le d\min(s_i, t_j)$ for any $u_i, v_j \in \{-1, 0, +1\}^n$. We now bound the two terms above. For the first term,

$$\begin{aligned}
\left|\sum_{(i,j)\in[r]\times[\ell]:j\geq i+\frac{1}{2}\log d} 2^{-i-j}u_i^T A_s v_j\right| &\leq \sum_{i\in[r]}\sum_{j=i+\frac{1}{2}\log d}^{\ell} 2^{-i-j}|u_i^T A_s v_j| \\
&\leq \sum_{i\in[r]}\sum_{j=i+\frac{1}{2}\log d}^{\ell} 2^{-i-j} d\cdot\min(s_i, t_j) \\
&\leq \sum_{i\in[r]}\sum_{j=i+\frac{1}{2}\log d}^{\ell} 2^{-i-j} d\cdot s_i \\
&\leq 2\sqrt{d}\sum_{i\in[r]} 2^{-2i}s_i.
\end{aligned}$$

For the second term,

$$\begin{aligned}
\left|\sum_{\substack{(i,j)\in[r]\times[\ell]:i\leq j<i+\frac{1}{2}\log d,\\ \max(s_i,t_j)\geq d\min(s_i,t_j)}} 2^{-i-j}u_i^T A_s v_j\right| &\leq \sum_{\substack{i\in[r],j\in[\ell]:i\leq j<i+\frac{1}{2}\log d,\\ \max(s_i,t_j)\geq d\min(s_i,t_j)}} 2^{-i-j}|u_i^T A_s v_j| \\
&\leq \sum_{\substack{i\in[r],j\in[\ell]:i\leq j<i+\frac{1}{2}\log d,\\ \max(s_i,t_j)\geq d\min(s_i,t_j)}} 2^{-i-j} d\min(s_i, t_j) \\
&\leq \sum_{\substack{i\in[r],j\in[\ell]:i\leq j<i+\frac{1}{2}\log d,\\ \max(s_i,t_j)\geq d\min(s_i,t_j)}} 2^{-i-j} max(s_i, t_j) \\
&\leq \sum_{\substack{i\in[r],j\in[\ell]:i\leq j<i+\frac{1}{2}\log d,\\ \max(s_i,t_j)\geq d\min(s_i,t_j)}} 2^{-i-j}(s_i + t_j) \\
&= \sum_{i\in[r]} 2^{-i}s_i \sum_{j=i}^{i+\frac{1}{2}\log d} 2^{-j} + \sum_{j\in[\ell]} 2^{-j}t_j \sum_{i=j-\frac{1}{2}\log d}^{i=j} 2^{-i} \\
&\leq \frac{2}{\sqrt{d}}\sum_{i\in[r]} s_i 2^{-2i} + 2\sqrt{d}\sum_{j\in[\ell]} t_j 2^{-2j}.
\end{aligned}$$

□

Claim 5.

$$\left|\sum_{(i,j)\in C_2}(2^{-i}u_i^T)A_s(2^{-j}v_j)\right| \leq 28\max\left(\sqrt{d}, \sqrt{\lambda\log d}\right)\sum_{i\in[r]} s_i 2^{-2i}.$$

Proof. The sum is conditioned over the set of tuples (i, j) in C_2, where

$$C_2 = \{(i,j)\in[r]\times[\ell]|(i\leq j < i + \frac{1}{2}\log d) \text{ and } (t_j \leq s_i < d\cdot t_j)\}.$$

By triangle inequality the required sum is at most $\sum_{(i,j)\in C_2} 2^{-i-j}|u_i^T A_s v_j|$. We note that $u_i, v_j \neq \overline{0}$ since $t_j \leq s_i < dt_j$. Consider the term $|u_i^T A_s v_j|$ where (i, j) is in C_2. We have two cases:
Case 1: If $(d/\lambda)\sqrt{s_i t_j} \geq n$, then we use Lemma 4 for the choice $a \leftarrow u_i, b_0 \leftarrow v_j$. This choice satisfies the

conditions of Lemma 4. Hence,

$$|u_i^T A_s v_j| \le 14\sqrt{d \cdot s_i^2 \cdot \frac{t_j}{n} \log\left(\frac{2n}{t_j}\right)} \le 14\sqrt{d} s_i.$$

Here, the last inequality follows by using $x \log\left(\frac{2}{x}\right) \le 1$ for $x < 1$.

Case 2: If $(d/\lambda)\sqrt{s_i t_j} < n$, then we use Corollary 3 for the choice $a \leftarrow v_j, b \leftarrow u_i$. This choice satisfies the conditions of Corollary 3 since $t_j \le s_i < dt_j$, condition (I) of the Lemma implies $s_i > n/d^2$, and $(d/\lambda)\sqrt{s_i t_j} < n$. Hence,

$$|u_i^T A_s v_j| \le 14\sqrt{\lambda \sqrt{t_j s_i} s_i \log\left(\frac{2 \cdot d \cdot t_j}{s_i}\right)} \le 14\sqrt{\lambda \log d} s_i.$$

The last inequality follows since $t_j \le s_i$.

Thus, for $(i,j) \in C_2$, we have $|u_i^T A_s v_j| \le 14 \max(\sqrt{d}, \sqrt{\lambda \log d}) s_i$. Therefore,

$$\begin{aligned} \left| \sum_{(i,j) \in C_2} (2^{-i} u_i^T) A_s (2^{-j} v_j) \right| &\le \sum_{(i,j) \in C_2} 2^{-i-j} |u_i^T A_s v_j| \\ &\le 14 \sum_{i \in [r]} \sum_{j=i}^{\infty} 2^{-i-j} \max(\sqrt{d}, \sqrt{\lambda \log d}) s_i \\ &\le 28 \max\left(\sqrt{d}, \sqrt{\lambda \log d}\right) \sum_{i \in [r]} s_i 2^{-2i}. \end{aligned}$$

□

Claim 6.

$$\left| \sum_{(i,j) \in C_3} (2^{-i} u_i^T) A_s (2^{-j} v_j) \right| = \left(\frac{\lambda}{5} + 0.95 \cdot 10^{12} \sqrt{d}\right) \sum_{j \in [\ell]} t_j 2^{-2j}.$$

Proof. The sum is conditioned over the set of tuples (i,j) in C_3, where

$$C_3 = \left\{ (i,j) | (i \le j \le i + \frac{1}{2}\log d) \wedge (s_i \le t_j < ds_i) \wedge \left(\frac{d}{\lambda}\sqrt{s_i t_j} < n\right) \wedge \left(s_i 2^{-2i} < \frac{\lambda}{\sqrt{d}} t_j 2^{-2j}\right) \right\}.$$

By triangle inequality,

$$\left| \sum_{(i,j) \in C_3} (2^{-i} u_i^T) A_s (2^{-j} v_j) \right| \le \sum_{(i,j) \in C_3} 2^{-i-j} |u_i^T A_s v_j|.$$

We note that $u_i, v_j \ne \overline{0}$ since $s_i \le t_j < ds_i$. We use Corollary 3 to bound each term $|u_i^T A_s v_j|$. We use Corollary 3 with the choice $a \leftarrow u_i$ and $b \leftarrow v_j$. This choice satisfies the conditions of Corollary 3 since

$s_i \le t_j \le ds_i$, condition (I) of the Lemma implies $t_j > n/d^2$, and $(d/\lambda)\sqrt{s_it_j} < n$. Hence,

$$\begin{aligned}
\sum_{(i,j)\in C_3} 2^{-i-j}|u_i^T A_s v_j| &\le 10 \sum_{(i,j)\in C_3} 2^{-i-j}\sqrt{\lambda\sqrt{s_it_j}t_j \log\left(\frac{2ds_i}{t_j}\right)} \\
&< 10 \sum_{(i,j)\in C_3} \frac{(\lambda)^{3/4}}{d^{1/8}} t_j 2^{-i-j}\sqrt{2^{-(j-i)}\log\left(\frac{2\lambda\sqrt{d}}{2^{2j-2i}}\right)} && \left(s_i 2^{-2i} < \frac{\lambda}{\sqrt{d}} t_j 2^{-2j}\right) \\
&\le 10\frac{(\lambda)^{3/4}}{d^{1/8}} \sum_{j\in[\ell]} t_j 2^{-2j} \sum_{i=j-\frac{1}{2}\log d+1}^{i=j} \sqrt{2^{j-i}\log\left(\frac{2\lambda\sqrt{d}}{2^{2j-2i}}\right)} \\
&= 90\frac{\lambda^{3/4}}{d^{1/8}}\sqrt{\sqrt{d}\log\left(\frac{2\lambda}{\sqrt{d}}\right)} \sum_{j\in[\ell]} t_j 2^{-2j} && (\text{by Lemma 2 and } \lambda \ge \sqrt{d}) \\
&= 90\lambda\sqrt{\sqrt{\frac{\sqrt{d}}{\lambda}}\log\left(\frac{2\lambda}{\sqrt{d}}\right)} \sum_{j\in[\ell]} t_j 2^{-2j}.
\end{aligned}$$

By Fact 1, (we can chose an appropriate constant c_1) such that the above quantity is bounded by

$$\left(\frac{\lambda}{5} + 0.95\cdot 10^{12}\sqrt{d}\right)\sum_{j\in[\ell]} t_j 2^{-2j}.$$

□

Claim 7.

$$\left|\sum_{(i,j)\in C_4} (2^{-i}u_i^T)A_s(2^{-j}v_j)\right| = 136\sqrt{d}\sum_{i\in[r]} s_i 2^{-2i}.$$

Proof. The sum is conditioned over the set of tuples (i,j) in C_4, where

$$C_4 = \left\{(i,j)|(i \le j < i + \frac{1}{2}\log d) \wedge (s_i \le t_j < ds_i) \wedge \left(\frac{d}{\lambda}\sqrt{s_it_j} < n\right) \wedge \left(s_i 2^{-2i} \ge \frac{\lambda}{\sqrt{d}} t_j 2^{-2j}\right)\right\}.$$

By triangle inequality,

$$\left|\sum_{(i,j)\in C_4} (2^{-i}u_i^T)A_s(2^{-j}v_j)\right| \le \sum_{(i,j)\in C_4} 2^{-i-j}|u_i^T A_s v_j|.$$

We note that $u_i, v_j \ne \overline{0}$ since $s_i \le t_j < ds_i$. We use Corollary 3 to bound each term $|u_i^T A_s v_j|$. We use Corollary 3 with the choice $a \leftarrow u_i$ and $b \leftarrow v_j$. This choice satisfies the conditions of Corollary 3 since

$s_i \le t_j \le ds_i$, condition (I) of the Lemma implies $t_j > n/d^2$, and $(d/\lambda)\sqrt{s_i t_j} < n$. Hence,

$$\begin{aligned}
\left|\sum_{(i,j)\in C_4} (2^{-i}u_i^T)A_s(2^{-j}v_j)\right| &\le \sum_{(i,j)\in C_4} 2^{-i-j}|u_i^T A_s v_j| \\
&\le 10 \sum_{(i,j)\in C_4} 2^{-i-j}\sqrt{\lambda\sqrt{s_i t_j}t_j \log\left(\frac{2ds_i}{t_j}\right)} \\
&= 10 \sum_{(i,j)\in C_4} 2^{-i-j}\sqrt{\lambda}s_i\sqrt{\left(\frac{t_j}{s_i}\right)^{\frac{3}{2}} \log\left(\frac{2ds_i}{t_j}\right)} \\
&\le 10 \sum_{(i,j)\in C_4} 2^{-i-j}\frac{d^{3/8}}{\lambda^{1/4}}s_i\sqrt{2^{3j-3i}\log\left(\frac{2\lambda\sqrt{d}}{2^{2j-2i}}\right)}
\end{aligned}$$

Above we use the fact that $x^{\frac{3}{2}}\log\left(\frac{c}{x}\right)$ is an increasoing function if $x \le \frac{c}{2}$ and $s_i 2^{-2j} \ge \frac{\lambda}{\sqrt{d}}t_j 2^{-2j}$. Therefore,

$$\begin{aligned}
\left|\sum_{(i,j)\in C_4} (2^{-i}u_i^T)A_s(2^{-j}v_j)\right| &\le 10\sum_{i\in[r]} \frac{d^{3/8}}{\lambda^{1/4}}s_i 2^{-2i} \sum_{j=i}^{j=i+\frac{1}{2}\log d-1} \sqrt{2^{j-i}2\log\left(\frac{2\lambda\sqrt{d}}{2^{2j-2i}}\right)} \\
&= 90\sum_{i\in[r]} \frac{d^{3/8}}{\lambda^{1/4}}s_i 2^{-2i}\sqrt{\sqrt{d}\log\left(\frac{2\lambda}{\sqrt{d}}\right)} && \text{(by Lemma 2)} \\
&= 90\sum_{i\in[r]} d^{\frac{1}{2}}s_i 2^{-2i}\sqrt{\sqrt{\frac{\sqrt{d}}{\lambda}}\log\left(\frac{2\lambda}{\sqrt{d}}\right)} \\
&= 136\sqrt{d}\sum_{i\in[r]} s_i 2^{-2i}.
\end{aligned}$$

That last equality is because, $\lambda \ge \sqrt{d}$ for every d-regular graph and hence $\sqrt{\frac{\sqrt{d}}{\lambda}}\log\left(\frac{2\lambda}{\sqrt{d}}\right) \le 1.502$. ☐

Claim 8.

$$\left|\sum_{(i,j)\in C_5} (2^{-i}u_i^T)A_s(2^{-j}v_j)\right| \le 56\sqrt{d}\left(\sum_{j\in[l]} t_j 2^{-2j} + \sum_{i\in[r]} s_i 2^{-2i}\right).$$

Proof. The sum is conditioned over the set of tuples (i,j) in C_5, where

$$C_5 = \left\{(i,j)|(i \le j < i + \frac{1}{2}\log d) \wedge (s_i \le t_j < ds_i) \wedge \left(\frac{d}{\lambda}\sqrt{s_i t_j} \ge n\right) \wedge \left(s_i 2^{-2i} < t_j 2^{-2j}\right)\right\}.$$

By triangle inequality,

$$\left|\sum_{(i,j)\in C_5} (2^{-i}u_i^T)A_s(2^{-j}v_j)\right| \le \sum_{j\in[\ell]:\exists i\in[r] \text{ with } (i,j)\in C_5} 2^{-2j}\left|\sum_{i:(i,j)\in C_5} 2^{-i+j}u_i^T A_s v_j\right|.$$

We note that $u_i, v_j \ne \overline{0}$ since $s_i \le t_j < ds_i$ for every $(i,j) \in C_5$. Let us fix j such that there exists $(i,j) \in C_5$. We bound

$$\left|\sum_{\substack{i\in\{j-(1/2)\log d,\dots,j\}:\\ (i,j)\in C_5}} 2^{-i+j}u_i^T A_s v_j\right|$$

using Lemma 4. We will use Lemma 4 for the choice $a \leftarrow v_j$ and for every $k = 0, 1, \ldots, (1/2)\log d$, we take $b_k \leftarrow u_{j-k}$ if $(j-k, j) \in C_5$ and $b_k \leftarrow \overline{0}$ if $(j-k, j) \notin C_5$. This choice satisfies the conditions of Lemma 4 since (i) condition (I) of the Lemma implies $S(b_k)$ are mutually non-intersecting, (ii) $|S(v_j)| = t_j \geq 2^{2j-2i}s_i = 2^{2j-2i}|S(u_i)|$ for every $(i,j) \in C_5$ implies $|S(a)| \geq 2^{2k}|S(b_k)|$ for every $k = 0, 1, \ldots, (1/2)\log d$, and (iii) b_k is non-zero if and only if $(j-k, j) \in C_5$ implies $(d/\lambda)\sqrt{|S(b_k)||S(a)|} \geq n$ for every non-zero b_k. Hence, by Lemma 4, we have

$$\sum_{j \in [\ell]} 2^{-2j} \left| \sum_{i:(i,j)\in C_5} 2^{-i+j} u_i^T A_s v_j \right| \leq 14 \sum_{j\in[\ell]} 2^{-2j} \sqrt{\frac{dt_j^2}{n} \sum_{i=j-\frac{1}{2}\log d}^{i=j} s_i 2^{-2i+2j} \log\left(\frac{2n}{t_j}\right)}$$
$$= 14\sqrt{d} \sum_{j\in[\ell]} \sqrt{2^{-2j}\frac{t_j^2}{n} \log\left(\frac{2n}{t_j}\right) \sum_{i=j-\frac{1}{2}\log d}^{i=j} s_i 2^{-2i}}.$$

Next, we group v_j according to their support sizes and then sum them together. For $c = 0, 1, 2, \ldots, \log(n)$, let J_c be the set of indices $j \in [\ell]$ s.t. $n/2^c \leq t_j < 2n/2^c$ and for non-empty sets J_c, define $j_c := \min(j \in J_c)$. With this notation, the above sum is

$$\leq 14\sqrt{d} \sum_{c=0}^{\log n} \sum_{j\in J_c} \sqrt{4n 2^{-2j-2c} \log(2\cdot 2^c) \sum_{i=j-1/2\log d+1}^{i=j} s_i 2^{-2i}} \qquad \left(\frac{n}{2^c} \leq t_j < \frac{2n}{2^c}\right)$$
$$\leq 14\sqrt{d} \sum_{c=0}^{\log n} \sum_{j\in J_c} \frac{1}{2}\left(4n2^{-j-j_c-c} + 2^{-j+j_c-c}\log(2\cdot 2^c) \sum_{i=j-1/2\log d+1}^{i=j} s_i 2^{-2i}\right) \qquad (G.M. \leq A.M.)$$
$$= 28\sqrt{d} \sum_{c=0}^{\log n} \sum_{j\in J_c} n2^{-j-j_c-c} + 7\sqrt{d} \sum_{c=0}^{\log n} \sum_{j\in J_c} \sum_{i=j-1/2\log d+1}^{i=j} 2^{-j+j_c-c} \log(2\cdot 2^c) s_i 2^{-2i}$$
$$\leq 28\sqrt{d} \sum_{c=0}^{\log n} \sum_{j\in J_c} \frac{n}{2^c} 2^{-j-j_c} + 7\sqrt{d} \sum_{i\in[r]} s_i 2^{-2i} \sum_{c=0}^{\log n} \frac{\log(2\cdot 2^c)}{2^c} \sum_{j\in J_c} 2^{-j+j_c}.$$

We observe that

$$\sum_{c=0}^{\log n} \sum_{j\in J_c} \frac{n}{2^c} 2^{-j-j_c} \leq \sum_{c=0}^{\log n} \sum_{j\in J_c} t_j 2^{-j-j_c} \leq \sum_{c=0}^{\log n} \sum_{j\in J_c} t_j 2^{-2j} = \sum_{j\in[\ell]} t_j 2^{-2j}.$$

Moreover, $\sum_{j\in J_c} 2^{-j+j_c} \leq 2$ and $\sum_{c=0}^{\log n} \frac{\log(2.2^c)}{2^c} \leq 4$. Substituting these we have the claim. □

Claim 9.

$$\left| \sum_{(i,j)\in C_6} (2^{-i}u_i^T) A_s (2^{-j} v_j) \right| = 154\sqrt{d} \sum_{i\in[r]} s_i 2^{-2i}.$$

Proof. The sum is conditioned over the set of tuples (i, j) in C_6, where

$$C_6 = \left\{ (i,j) | (i \leq j \leq i + \frac{1}{2}\log d) \wedge (s_i \leq t_j < ds_i) \wedge (\frac{d}{\lambda}\sqrt{s_i t_j} \geq n) \wedge \left(s_i 2^{-2i} \geq t_j 2^{-2j}\right) \right\}.$$

By triangle inequality,

$$\left| \sum_{(i,j)\in C_6} (2^{-i}u_i^T)A_s(2^{-j}v_j) \right| \leq \sum_{(i,j)\in C_6} 2^{-i-j}|u_i^T A_s v_j|.$$

We will use Lemma 4 to bound each term $|u_i^T A_s v_j|$. We use Lemma 4 with the choice $a \leftarrow v_j, b_0 \leftarrow u_i$. This choice satisfies the conditions of Lemma 4 since $s_i \leq t_j < ds_i$ and $(d/\lambda)\sqrt{s_i t_j} \geq n$. Hence,

$$\sum_{(i,j)\in C_6} 2^{-i-j}|u_i^T A_s v_j| \leq 14 \sum_{(i,j)\in C_6} 2^{-i-j} \sqrt{\frac{ds_i t_j^2}{n} \log\left(\frac{2n}{t_j}\right)}.$$

Next, we divide the tuples in C_6 into two parts depending on the value of i and j:

$$C_6' := \{(i,j)|(i,j) \in C_6, (i \leq j < i + \frac{1}{2}\log(n/s_i))\} \text{ and}$$
$$C_6'' := \{(i,j)|(i,j) \in C_6, (j \geq i + \frac{1}{2}\log(n/s_i))\}.$$

Let us consider the above RHS sum over tuples (i,j) in C_6'.

$$\begin{aligned}
14 \sum_{(i,j)\in C_6'} 2^{-i-j} \sqrt{\frac{ds_i t_j^2}{n} \log\left(\frac{2n}{t_j}\right)} &= 14\sqrt{d} \sum_{(i,j)\in C_6'} 2^{-2i} s_i \sqrt{2^{-2j+2i} \frac{1}{ns_i} t_j^2 \log\left(\frac{2n}{t_j}\right)} \\
&\leq 14\sqrt{d} \sum_{(i,j)\in C_6'} s_i 2^{-2i} \sqrt{\frac{s_i 2^{2j-2i}}{n} \log\left(\frac{2n}{s_i 2^{2j-2i}}\right)} \qquad (t_j 2^{-2j} \leq s_i 2^{-2i}) \\
&\leq 14\sqrt{d} \sum_{i\in[r]} s_i 2^{-2i} \sum_{j=i}^{j=i+\frac{1}{2}\log(\frac{n}{s_i})} \sqrt{\frac{s_i 2^{2j-2i}}{n} \log\left(\frac{2n}{s_i 2^{2j-2i}}\right)} \\
&\leq 126\sqrt{d} \sum_{i\in[r]} s_i 2^{-2i}.
\end{aligned}$$

In the above, the last inequality is by using Lemma 2 for $\sum_{j=i}^{j=i+\frac{1}{2}\log(\frac{n}{s_i})} \sqrt{\frac{s_i 2^{2j-2i}}{n} \log\left(\frac{2n}{s_i 2^{2j-2i}}\right)}$. Next, let us consider the RHS sum over tuples (i,j) in C_6''.

$$\begin{aligned}
14 \sum_{(i,j)\in C_6''} 2^{-i-j} \sqrt{\frac{ds_i t_j^2}{n} \log\left(\frac{2n}{t_j}\right)} &= 14\sqrt{d} \sum_{(i,j)\in C_6''} s_i 2^{-i-j} \sqrt{\frac{t_j}{s_i}} \sqrt{\frac{t_j}{n} \log\left(\frac{2n}{t_j}\right)} \\
&\leq 14\sqrt{d}) \sum_{(i,j)\in C_6''} s_i 2^{-i-j} \sqrt{\frac{n}{s_i}} \qquad \left(t_j \leq n, x\log\frac{2}{x} \leq 1\right) \\
&\leq 14\sqrt{d} \sum_{i\in[r]} s_i 2^{-2i} \sum_{j=i+\frac{1}{2}\log(n/s_i)}^{\infty} 2^{-j+i} \sqrt{\frac{n}{s_i}} \\
&\leq 28\sqrt{d} \sum_{i\in[r]} s_i 2^{-2i}.
\end{aligned}$$

The claim follows from the above two bounds. $\square$

We now obtain the required bound for conclusion 1 of the Lemma from Claims 4, 5, 6, 7, 8, and 9:

$$\left|\sum_{(i,j)\in[r]\times[\ell]}(2^{-i}u_i^T)A_s(2^{-j}v_j)\right| \leq 377\max\left(\sqrt{\lambda\log d},\sqrt{d}\right)\sum_{i\in[r]}s_i 2^{-2i}+\left(\frac{\lambda}{5}+10^{12}\sqrt{d}\right)\sum_{j\in[\ell]}t_j 2^{-2j}.$$

For conclusion 2 of the Lemma, we observe that if $s_i \geq t_j$ for all $i \in [r], j \in [\ell]$, then C_3, C_4, C_5, C_6 are empty. Thus the bound follows from Claims 4 and 5:

$$\left|\sum_{(i,j)\in[r]\times[\ell]}(2^{-i}u_i^T)A_s(2^{-j}v_j)\right| \quad\leq\quad 31\max\left(\sqrt{\lambda\log d},\sqrt{d}\right)\left(\sum_{i\in[r]}s_i 2^{-2i}+\sum_{j\in[l]}t_j 2^{-2j}\right).$$

□

6 Proofs of Theorems 1 and 2

We will prove Theorem 2. Theorem 1 follows as a special case. To prove Theorem 2, we need the following modified version of Lemma 3.

Lemma 6. *Let G be a d -regular graph with non-trivial eigenvalues at most λ in absolute value $\sqrt{d} \leq \lambda, 2 \leq d \leq \sqrt{\frac{n}{3\ln n}}$ and let A be the adjacency matrix of G. Let A' be a random $n \times n$ real matrix whose entries $A'(i,j)$ are random variables with mean 0, $|A'(i,j)| \leq A(i,j)$ for all i,j, and the entries $A'(i,j)$ are independent from all other entries except $A'(j,i)$. There exist constants $c_1, c_2 \geq 1000, c_3, c_4$ such that the following statements hold with probability at least $1 - e^{-(n/d^2)}$ (over the random choice of A').*

1. *For all $u_1, u_2, \ldots u_r \in \{0, \pm 1, \pm\frac{1}{2}\}^n$, $v_1, v_2 \ldots, v_\ell \in \{0, \pm 1, \pm\frac{1}{2}\}^n$ satisfying*

   (I) $S(u_i) \cap S(u_j) = \phi$ for every $i, j \in [r]$ and $S(v_i) \cap S(v_j) = \phi$ for every $i, j \in [\ell]$, and

   (II) Either $|S(u_i)| > n/d^2$ for every $i \in [r]$ with non-zero u_i, or $|S(v_i)| > n/d^2$ for every $i \in [\ell]$ with non-zero v_i,

   we have

   $$\left|\sum_{i\leq j}(2^{-i}u_i^T)A'(2^{-j}v_j)\right| \leq c_1\max\left(\sqrt{\lambda\log d},\sqrt{d}\right)\sum_{i=1}^{r}|S(u_i)|2^{-2i}+\left(\frac{\lambda}{c_2}+c_3\sqrt{d}\right)\sum_{j=1}^{\ell}|S(v_j)|2^{-2j}$$

2. *For all $u_1, u_2, \ldots u_r \in \{0, \pm 1, \pm\frac{1}{2}\}^n$, $v_1, v_2 \ldots, v_\ell \in \{0, \pm 1, \pm\frac{1}{2}\}^n$ satisfying (I), (II) and*

   (III) $|S(u_i)| > |S(v_j)|$ for every $i \in [r], j \in [\ell]$ with non-zero u_i,

   we have

   $$\left|\sum_{i\leq j}(2^{-i}u_i^T)A'(2^{-j}v_j)\right| \leq c_4\max\left(\sqrt{\lambda\log d},\sqrt{d}\right)\left(\sum_{i=1}^{r}|S(u_i)|2^{-2i}+\sum_{j=1}^{\ell}|S(v_j)|2^{-2j}\right).$$

The proof of Lemma 6 is identical to that of Lemma 3. In the proof of Lemma 3, we used the concentration inequalities from Lemma 4 and Corollary 3. We note that these concentration inequalities were obtained using Hoeffding's inequality. Since Hoeffding's inequality is applicable when the random variables are bounded, we have the version of Lemma 4 and Corollary 3 applicable to the random matrix A'. As a consequence, we obtain Lemma 6 by following the same proof strategy as that of Lemma 3. We avoid repeating the proof for brevity.

Proof of Theorem 2. The proof is very similar to the proof of Theorem **??**. However, in order to avoid a loss of factor 4, we avoid discretizing in the first step, but discretize only for certain cases. Using Lemma 1, we know that for a shift k-lift, λ_{new} is the maximum absolute value in the set

$$\bigcup_{\omega:\ \omega \text{ is a } k\text{-th primitive root of unity, } \omega\neq 1} \text{eigenvalues}\left(A_s(\omega)\right).$$

We will bound the probability that the maximum eigenvalue of $A_s(\omega)$ is large for ω being a fixed primitive kth root of unity. A union bound over the $k-1$ primitive k-th roots of unity bounds the maximum eigenvalues of all $k-1$ matrices simultaneously.

Let us fix ω to be a primitive kth root of unity and bound the eigenvalues of $A_s(\omega)$. We need to bound $\max_{x\in\mathbb{C}^n} \frac{|x^* A_s(\omega) x|}{|x^* x|}$ where x^* denotes the complex conjugate of vector x. Let $x = q + iw \in C^n$ where $q, w \in \mathbb{R}^n$. We consider a decomposition of q, w (similar to but not same as diadic decomposition) into a sequence of vectors y_i's and z_i's for $i \in \{0, 1, \ldots,\}$ respectively as follows:

$$[y_i]_j := \begin{cases} q_j & \text{if } 2^{-i-1} < |q_j| \leq 2^{-i}, \\ 0 & \text{otherwise,} \end{cases}$$

$$[z_i]_j := \begin{cases} w_j & \text{if } 2^{-i-1} < |w_j| \leq 2^{-i}, \\ 0 & \text{otherwise.} \end{cases}$$

Let us partition the set of indices $\{0, 1, \ldots\}$ into two sets $M_r := \{i : |S(y_i)| < n/d^2\}$ and $L_r := \{i : |S(y_i)| \geq n/d^2\}$ and define $y_{M_r} := \sum_{i\in M_r} y_i$ and $y_{L_r} := \sum_{i\in L_r} y_i$. Similarly, define M_c and L_c based on z_i and define z_{M_c} and z_{L_c}. We will refer to vectors y_{M_r}, z_{M_c} as "type M" vectors, and y_{L_r} and z_{L_c} as "type L" vectors. We note that

$$x^* x = ||y_{M_r}||^2 + ||y_{L_r}||^2 + ||z_{M_c}||^2 + ||z_{L_c}||^2.$$

By splitting the terms in $|x^* A_s(\omega) x|$, we get

$$\begin{aligned} |x^* A_s(\omega) x| \leq |(y_{M_r} + iz_{M_c})^* A_s(\omega)(y_{M_r} + iz_{M_c})| + |z_{L_c}^T A_s(\omega) y_{L_r}| + |y_{L_r}^T A_s(\omega) z_{L_c}| \\ + |y_{L_r}^T A_s(\omega) y_{L_r}| + |y_{L_r}^T A_s(\omega) y_{M_r}| + |y_{M_r}^T A_s(\omega) y_{L_r}| \\ + |z_{L_c}^T A_s(\omega) z_{L_c}| + |z_{L_c}^T A_s(\omega) z_{M_c}| + |z_{M_c}^T A_s(\omega) z_{L_c}| \\ + |y_{L_r}^T A_s(\omega) z_{M_c}| + |z_{M_c}^T A_s(\omega) y_{L_r}| + |z_{L_c}^T A_s(\omega) y_{M_r}| + |y_{M_r}^T A_s(\omega) z_{L_c}| \end{aligned} \tag{3}$$

To derive an upper bound on $||x^* A_s(\omega) x||$, we will show upper bounds for each of the terms in the RHS using Lemma 6. We note that the concentration inequalities given in parts 1 and 2 of Lemma 6 hold with probability at least $1 - e^{-n/d^2}$ for some constants $c_1, c_2 \geq 1000, c_3, c_4$. Assuming parts 1 and 2 of Lemma 6, we have the following claims:

Claim 10.

$$|(y_{M_r} + iz_{M_c})^* A_s(\omega)(y_{M_r} + iz_{M_c})| \leq \left(\lambda + \frac{64}{d}\right) ||y_{M_r} + iz_{M_c}||^2.$$

Claim 11. *For any type L vectors a and b,*

$$|a^T A_s(\omega) b| \leq \left(\frac{32\lambda}{c_2} + 32(c_1 + c_3)\left(\max\left(\sqrt{\lambda \log(d)}, \sqrt{d}\right)\right)\right)\left(||a||^2 + ||b||^2\right).$$

Claim 12. *For any vector a of type M and vector b of type L,*

$$|a^T A_s(\omega) b| \leq \frac{32\lambda}{c_2}||b||^2 + 32(c_1 + c_3 + c_4)\left(\max\left(\sqrt{\lambda \log(d)}, \sqrt{d}\right)\right)\left(||b||^2 + ||a||^2\right).$$

We note that all terms in the RHS of inequality (3) fall into one of three categories in Claims 10, 11 and

12 above. Using these bounds, the following holds with probability at least $1 - e^{-(n/d^2)}$:

$$\begin{aligned}
|x^* A_s(\omega)x| &\leq \left(\lambda + \frac{64}{d}\right)||y_{M_r} + iz_{M_c}||^2 + \frac{256\lambda}{c_2}\left(||y_{L_r}||^2 + ||z_{L_c}||^2\right) \\
&\quad + 256(c_1 + c_3 + c_4)\left(\max\left(\sqrt{\lambda \log(d)}, \sqrt{d}\right)\right)\left(||y_{M_r}||^2 + ||z_{M_c}||^2 + ||y_{L_r}||^2 + ||z_{L_c}||^2\right) \\
&\leq \left(\lambda + 288(c_1 + c_3 + c_4)\left(\max\left(\sqrt{\lambda \log(d)}, \sqrt{d}\right)\right)\right)x^* x.
\end{aligned}$$

The last inequality is because $c_2 \geq 1000$ and $d \geq 2$. Taking a union bound over the k primitive roots of unity shows that there exists a constant c such that with probability at least $1 - ke^{-(n/d^2)}$, all new eigenvalues of a random shift k-lift have absolute value at most

$$\lambda + c \max\left(\sqrt{\lambda \log(d)}, \sqrt{d}\right).$$

$\square$

Proof of Claim 10. We observe that $|(y_M + iz_M)^* A_s(\omega)(y_M + iz_M)| \leq y'^T Ay'$ where y' is such that j-th coordinate of y' is equal to norm of the j-th element in $(y_M + iz_M)$ and A is the adjacency matrix of the base graph. Let J be a $n \times n$ matrix with all entries being 1. Then $y'^T Ay' = y'^T(A - \frac{d}{n}J)y' + y'^T(\frac{d}{n}J)y'$. The maximum eigenvalue of $A - (\frac{d}{n}J)$ is λ. Hence, $y'^T(A - \frac{d}{n}J)y' \leq \lambda||y'||^2 = \lambda||(y_M + iy_M)||^2$.

It remains to bound $y'^T \frac{d}{n} Jy'$. Let y'_{M_r} and z'_{M_c} be vectors obtained by taking the absolute values of the coordinates of y_{M_r} and z_{M_c} respectively. We have

$$y'^T\left(\frac{d}{n}J\right)y' \leq (y'_{M_r} + z'_{M_c})^T\left(\frac{d}{n}J\right)(y'_{M_r} + z'_{M_c}).$$

We recall that the number of entries between 2^{-i-1} and 2^{-i} in y'_{M_r} and z'_{M_c} are less than $\frac{n}{d^2}$. We will show that $|u^T(\frac{d}{n})Jv| \leq \frac{4}{d}(||u||^2 + ||v||^2)$ where $u, v \in \{y'_{M_r}, z'_{M_c}\}$.

Let $u, v \in \{y'_{M_r}, z'_{M_c}\}$. By Lemma 1, there exist u', v' s.t. $|u^T \frac{d}{n} Jv| \leq |u'^T \frac{d}{n} Jv'|$ where $u', v' \in \{0, \pm\frac{1}{2}, \pm\frac{1}{4}, \ldots\}^n$, $||u'||^2 \leq 4||u||^2$, and $||v'||^2 \leq 4||v||^2$. Consider the diadic decomposition of $u' = \sum_{i=0}^{\infty} 2^{-i}u_i$ obtained as follows: a coordinate of u_i is 1 if the corresponding coordinate of u' is 2^{-i}, it is -1 if the corresponding coordinate of u' is -2^{-i} and is 0 otherwise. Similarly, define the diadic decomposition of $v' = \sum_{j=0}^{\infty} 2^{-j}v_j$. We note that all entries between 2^{-i-1} and 2^{-i} in u and v are rounded to either 2^{-i-1} or 2^{-i} in u' and v' and all entries between -2^{-i-1} and -2^{i} are rounded to either -2^{-i-1} or -2^{-i}. And, since number of entries in u, v with absolute value between 2^{-i-1} and 2^{-i} is at most n/d^2, we get $|S(u_i)|, |S(v_j)| < \frac{2n}{d^2}$ for all i, j.

$$\begin{aligned}
\left|u'\left(\frac{d}{n}J\right)v'\right| &= \left|\sum_{i,j=0}^{\infty} 2^{-i-j}u_i^T\left(\frac{d}{n}J\right)v_j\right| \\
&\leq \sum_{i=0}^{\infty}\sum_{j=i}^{\infty} 2^{-i-j}\frac{d}{n}|u_i^T Jv_j| + \sum_{j=0}^{\infty}\sum_{i=j+1}^{\infty} 2^{-i-j}\frac{d}{n}|u_i^T Jv_j| \\
&\leq \sum_{i=0}^{\infty}\sum_{j=i}^{\infty} 2^{-i-j}\frac{d|S(u_i)||S(v_j)|}{n} + \sum_{j=0}^{\infty}\sum_{i=j+1}^{\infty} 2^{-i-j}\frac{d|S(v_j)||S(u_i)|}{n} \\
&\leq 2\sum_{i=0}^{\infty} 2^{-2i}\frac{|S(u_i)|}{d}\sum_{j=i}^{\infty} 2^{-j+i} + 2\sum_{j=0}^{\infty} 2^{-2j}\frac{|S(v_j)|}{d}\sum_{i=j+1}^{\infty} 2^{-i+j} \\
&\leq \frac{4}{d}\left(\sum_{i=0}^{\infty}|S(u_i)|2^{-2i} + \sum_{j=0}^{\infty}|S(v_j)|2^{-2j}\right) \\
&\leq \frac{4}{d}\left(||u'||^2 + ||v'||^2\right).
\end{aligned}$$

For $u,v \in \{y'_M, z'_M\}$, $u^T \frac{d}{n} J v \le u'^T \frac{d}{n} J v' \le \frac{4}{d}(||u'||^2 + ||v'||^2) \le \frac{16}{d}(||u||^2 + ||v||^2)$

$$\begin{aligned}
y'^T \frac{d}{n} J y' &\le (y'_{M_r} + z'_{M_c})^T \frac{d}{n} J (y'_{M_r} + z'_{M_c}) \\
&\le y'^T_{M_r} \frac{d}{n} J y'_{M_r} + y'^T_{M_r} \frac{d}{n} J z'_{M_c} + z'^T_{M_c} \frac{d}{n} J y'_{M_r} + z'^T_{M_c} \frac{d}{n} J z'_{M_c} \\
&\le \left(\frac{16}{d}\right) \left(||y'_{M_r}||^2 + ||y'_{M_r}||^2 + ||y'_{M_r}||^2 + ||z'_{M_c}||^2 + ||z'_{M_c}|| + ||y'_{M_r}||^2 + ||z'_{M_c}|| + ||z'_{M_c}||\right) \\
&= \left(\frac{64}{d}\right) \left(||y'_{M_r}||^2 + ||z'_{M_c}||^2\right) = \left(\frac{64}{d}\right) \left(||y_{M_r}||^2 + ||z_{M_c}||^2\right) = \left(\frac{64}{d}\right) ||y_{M_r} + i z_{M_c}||^2
\end{aligned}$$

Combining $y'^T(A - \frac{d}{n}J)y'$ and $y'^T \frac{d}{n} J y'$, we get $||(y_{M_r} + iz_{M_c})^* A_s(\omega)(y_{M_r} + iz_{M_c})|| \le y'Ay' \le (\lambda + (64/d))||(y_{M_r} + iz_{M_c})||^2$.

□

For Claims 11 and 12, we divide the matrix into its real and imaginary part: $A_s(\omega) = A^1_s(\omega) + iA^2_s(\omega)$ where $A^1_s(\omega)$ and $A^2_s(\omega)$ are real matrices. For any two vectors $a, b \in \mathbb{R}^n$,

$$|a^T A_s(\omega) b| \le |a^T A^1_s(\omega) b| + |a^T A^2_s(\omega) b|.$$

We will bound $|a^T A'_s(\omega) b|$ where $A'_s(\omega) \in \{A^1_s(\omega), A^2_s(\omega)\}$ for a, b as in Claims 11 and 12. We start by discretizing a and b. By Lemma 1, there exist a', b' such that $|a^T A'_s(\omega) b| \le |a'^T A'_s(\omega) b'|$ where $a', b' \in \{0, \pm\frac{1}{2}, \pm\frac{1}{4} \dots\}^n$ and $||a'||^2 \le 4||a||^2$ and $||b'||^2 \le 4||b||^2$. Moreover, every entry of a and b between 2^{-i-1} and 2^{-i} is rounded to either 2^{-i-1} or 2^{-i} in a' and b' respectively (similarly, every entry between -2^{-i-1} and -2^{-i} is rounded to either -2^{-i-1} or -2^{-i}). Consider the following vectors $\{u_i\}_{i \in \{0,1,\dots\}}$, $\{v_i\}_{i \in \{0,1,\dots\}}$ obtained from a', a and b, b' respectively:

$$[u_i]_j := \begin{cases} 2^i a'_j, & \text{if } 2^{-i-1} \le |a_j| < 2^{-i} \\ 0, & \text{otherwise} \end{cases}$$

$$[v_i]_j := \begin{cases} 2^i b'_j, & \text{if } 2^{-i-1} \le |b_j| < 2^{-i} \\ 0, & \text{otherwise} \end{cases}$$

We observe that $u_i, v_i \in \{0, \pm\frac{1}{2}, \pm 1\}^n$, $|a'^T A'_s(\omega) b'| = |\sum_{i,j=0}^{\infty} 2^{-i-j} u_i^T A'_s(\omega) v_j|$, $||a'||^2 = \sum_i 2^{-2i}||u_i||^2 \ge \frac{1}{4}\sum_i 2^{-2i}|S(u_i)|$ and

$$\left|\sum_{i,j=0}^{\infty} 2^{-i-j} u_i^T A'_s(\omega) v_j\right| \le \left|\sum_{i \le j} 2^{-i-j} u_i^T A'_s(\omega) v_j\right| + \left|\sum_{i<j} 2^{-i-j} v_i^T A'_s(\omega) u_j\right|.$$

Proof of Claim 11. Since a and b are type L vectors, we have $|S(u_i)|, |S(v_j)| \ge \frac{n}{d^2}$ for all non-zero u_i, v_j. By part 1 of Lemma 6,

$$\begin{aligned}
\left|\sum_{i \le j} 2^{-i-j} u_i^T A'_s(\omega) v_j\right| &\le c_1 \left(\max\left(\sqrt{\lambda \log(d)}, \sqrt{d}\right)\right) \sum_{i=0}^{\infty} |S(u_i)| 2^{-2i} + \left(\frac{\lambda}{c_2} + c_3\sqrt{d}\right) \sum_{j=0}^{\infty} |S(v_j)| 2^{-2j}, \\
\left|\sum_{i<j} 2^{-i-j} v_i^T A'_s(\omega) u_j\right| &\le c_1 \left(\max\left(\sqrt{\lambda \log(d)}, \sqrt{d}\right)\right) \sum_{i=0}^{\infty} |S(v_i)| 2^{-2i} + \left(\frac{\lambda}{c_2} + c_3\sqrt{d}\right) \sum_{j=0}^{\infty} |S(u_j)| 2^{-2j}.
\end{aligned}$$

Combining the above two we get

$$\begin{aligned}
\left|a'^T A'_s(\omega) b'\right| &= \left|\sum_{i,j=0}^{\infty} 2^{-i-j} u_i^T A_s(\omega) v_j\right| \\
&\leq \left(\frac{\lambda}{c_2} + (c_1+c_3)\left(\max\left(\sqrt{\lambda\log(d)}, \sqrt{d}\right)\right)\right)\left(\sum_{i=0}^{\infty} |S(u_i)| 2^{-2i} + \sum_{j=0}^{\infty} |S(v_j)2^{-2j}\right) \\
&\leq \left(\frac{4\lambda}{c_2} + 4(c_1+c_3)\left(\max\left(\sqrt{\lambda\log(d)}, \sqrt{d}\right)\right)\right)(||a'||^2 + ||b'||^2).
\end{aligned}$$

Hence,

$$\begin{aligned}
\left|a^T A'_s(\omega) b\right| &\leq |y^T A'_s(\omega) z| \leq \left(\frac{4\lambda}{c_2} + 4(c_1+c_3)\left(\max\left(\sqrt{\lambda\log(d)}, \sqrt{d}\right)\right)\right)\left(||a'||^2 + ||b'||^2\right) \\
&\leq \left(\frac{16\lambda}{c_2} + 16(c_1+c_3)\left(\max\left(\sqrt{\lambda\log(d)}, \sqrt{d}\right)\right)\right)\left(||a||^2 + ||b||^2\right).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\left|a^T A_s(\omega) b\right| &\leq |a^T A_s^1(\omega) b| + |a^T A_s^2(\omega) b| \\
&\leq \left(\frac{32\lambda}{c_2} + 32(c_1+c_3)\left(\max\left(\sqrt{\lambda\log(d)}, \sqrt{d}\right)\right)\right)\left(||a||^2 + ||b||^2\right).
\end{aligned}$$

□

Proof of Claim 12. Since, a is a vector of type M, b is a vector of type L, we have $|S(u_i)| < \frac{n}{d^2} \leq |S(v_j)|$ for all non-zero v_j. Applying parts 1 and 2 of Lemma 6, we get

$$\begin{aligned}
\left|\sum_{i\leq j} 2^{-i-j} u_i^T A'_s(\omega) v_j\right| &\leq c_1\left(\max\left(\sqrt{\lambda\log(d)}, \sqrt{d}\right)\right)\sum_{i=0}^{\infty} |S(u_i)| 2^{-2i} + \left(\frac{\lambda}{c_2} + c_3\sqrt{d}\right)\sum_{j=0}^{\infty} |S(v_j)| 2^{-2j}, \\
\left|\sum_{i<j} 2^{-i-j} v_i^T A'_s(\omega) u_j\right| &\leq c_4\left(\max\left(\sqrt{\lambda\log(d)}, \sqrt{d}\right)\right)\left(\sum_{i=0}^{\infty} |S(v_i)| 2^{-2i} + \sum_{j=0}^{\infty} |S(u_j)| 2^{-2j}\right).
\end{aligned}$$

Combining the above two, we get

$$\begin{aligned}
\left|a'^T A'_s(\omega) b'\right| &= \left|\sum_{i,j} 2^{-i-j} u_i^T A_s(\omega) v_j\right| \\
&\leq \frac{\lambda}{c_2}\sum_j |S(v_j)| 2^{-2j} \\
&\qquad + (c_1+c_3+c_4)\left(\max\left(\sqrt{\lambda\log(d)}, \sqrt{d}\right)\right)\left(\sum_j |S(v_j)| 2^{-2j} + \sum_i |S(u_i)| 2^{-2i}\right) \\
&\leq \frac{4\lambda}{c_2}\|b'\|^2 + 4(c_1+c_3+c_4)\left(\max\left(\sqrt{\lambda\log(d)}, \sqrt{d}\right)\right)\left(||b'||^2 + ||a'||^2\right).
\end{aligned}$$

Hence,

$$
\begin{aligned}
\left|a^T A_s'(\omega) b\right| &\leq \left|a'^T A_s'(\omega) b'\right| \leq \frac{4\lambda}{c_2}\|b'\|^2 + 4(c_1+c_3+c_4)\left(\max\left(\sqrt{\lambda\log(d)}, \sqrt{d}\right)\right)\left(||b'||^2 + ||a'||^2\right) \\
&\leq \frac{16\lambda}{c_2}\|b\|^2 + 16(c_1+c_3+c_4)\left(\max\left(\sqrt{\lambda\log(d)}, \sqrt{d}\right)\right)\left(||b||^2 + ||a||^2\right).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\left|a^T A_s(\omega) b\right| &\leq \left|a^T A_s^1(\omega) b\right| + \left|a^T A_s^2(\omega) b\right| \\
&\leq \frac{32\lambda}{c_2}\|b\|^2 + 32(c_1+c_3+c_4)\left(\max\left(\sqrt{\lambda\log(d)}, \sqrt{d}\right)\right)\left(||b||^2 + ||a||^2\right)
\end{aligned}
$$

□

References

[ABG10] L. Addario-Berry and S. Griffiths. The spectrum of random lifts. Arxiv preprint arXiv:1012.4097, 2010.

[Art98] M. Artin. *Algebra*. Birkhuser, 1998.

[BL06] Y. Bilu and N. Linial. Lifts, discrepancy and nearly optimal spectral gap. *Combinatorica*, 26(5):495–519, 2006.

[Chu89] F. R. K. Chung. Diameters and eigenvalues. *Journal of the American Mathematical Society*, 2(2):187–196, 1989.

[FKL04] R. Feng, J. H. Kwak, and J. Lee. Characteristic polynomials of graph coverings. *Bull. Austal. Math. Soc.*, 69:133–136, 2004.

[FMT06] J. Friedman, R. Murty, and J. Tillich. Spectral estimates for abelian cayley graphs. *J. Comb. Theory Ser. B*, 96(1):111–121, 2006.

[Fri03] J. Friedman. Relative expanders or weakly relatively ramanujan graphs. *Duke Math. J*, 118:2003, 2003.

[Fri08] J. Friedman. A proof of alon's second eiganvalue conjecture and related problems. *Mem. Amer. Math,Soc*, 195(910), 2008.

[Gre95] Y. Greenberg. On the spectrum of graphs and their universal coverings. Ph.D Thesis, 1995.

[HLW06] S. Hoory, N. Linial, and A. Wigderson. Expander graphs and their applications. *Bull. Amer. Math. Soc*, 43(4):439–561, 2006.

[HPS15] C. Hall, D. Puder, and W. Sawin. Ramanujan coverings of graphs, 2015.

[LP10] N. Linial and D. Puder. Word maps and spectra of random graph lifts. *Random Struct. Algorithms*, 37(1)):100–135, 2010.

[LPS88] A. Lubotzky, R. Phillips, and P. Sarnak. Ramanujan graphs. *Combinatorica*, 8(3):261–277, 1988.

[LSV11] E. Lubetzky, B. Sudakov, and V. Vu. Spectra of lifted ramanujan graphs. *Advances in Mathematics*, 227:16121645, 2011.

[LW03] N. Linial and A. Wigderson. Expander graphs and their applications, 2003.

[Mak15] A. Makelov. Expansion in lifts of graphs, 2015. Undergraduate Thesis, Harvard University.

[Mar88] G.A Margulis. Explicit group-theoretic constructions of combinatorial schemes and their applications in the construction of expanders and concentrators. *Probl. Inf. Transm*, 24(1):39–46, 1988.

[MS95] H. Mizuno and I. Sato. Characteristic polynomials of some graph coverings. *Discrete Mathematics*, 142:295–298, 1995.

[MSS13] A. Marcus, D. Spielman, and N. Srivastava. Interlacing families i: Ramanujan graphs of all degrees. In Proceedings, FOCS 2013, 2013.

[MSS15] A. Marcus, D. Spielman, and N. Srivastava. Interlacing families iv: Bipartite ramanujan graphs of all sizes, 2015. manuscript.

[Nil91] A. Nilli. On the second eigenvalue of a graph. *Discrete Math*, 91(2):207–210, 1991.

[Pin73] M. Pinsker. On the complexity of a concentrator. *7th International Teletraffic Conference*, pages 318/1–318/4, 1973.

[Pud13] D. Puder. Expansion of random graphs: New proofs, new results. Arxiv Preprint:1212.5216, 2013.

[Sar06] P. Sarnak. What is an expander? *Notices Amer. Math. Soc*, 51(7):762–763, 2006.

[Ser97] J. Serre. *Linear Representations of Finite Groups.* Springer, 1997.